Space groups for aperiodic crystals

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We report on the existence of symmetry plane–groups for quasiperiodic point-sets named beta–lattices. Like lattices are vector superpositions of integers, beta–lattices are vector superpositions of beta–integers. When $\beta > 1$ is a quadratic Pisot–Vijayaraghavan (PV) algebraic unit, the set of beta–integers can be equipped with an abelian group structure and an internal multiplicative law. When $\beta = (1+\sqrt{5})/2$, $1+\sqrt{2}$ and $2+\sqrt{3}$, we show that these arithmetic and algebraic structures lead to freely generated symmetry plane–groups for beta–lattices. These plane–groups are based on repetitions of discrete adapted rotations and translations we shall refer to as "beta–rotations" and "beta–translations". Hence beta–lattices, endowed with beta–rotations and beta–translations, can be viewed like lattices. We also show that, at large distances, beta–lattices and their symmetries behave asymptotically like lattices and lattice symmetries respectively.

PACS: 02.20.-a,61.44.-n
Key words: beta–lattices, Pisot numbers, quasicrystals, tilings, plane groups

1 Introduction: Lattices versus beta−lattices

After the discovery of modulated phases and of quasicrystals, Crystallography has been divided in two categories: periodic Crystallography, and aperiodic Crystallography [1]. Let us recall here the essential of the former.

- A crystallographic group in $\mathbb{R}^d$, or a space–group in $\mathbb{R}^d$, is a discrete group of isometries whose maximal translation subgroup is of rank d, hence isomorphic to $\mathbb{Z}^d$.
- A periodic crystal is the orbit under the action of a crystallographic group of a finite number of points of $\mathbb{R}^d$.

As an example, let us just consider the familiar square lattice of Figure 1 mathematically described by

$$\Lambda = \mathbb{Z} + \mathbb{Z}e^{i\pi/2}. \tag{1}$$

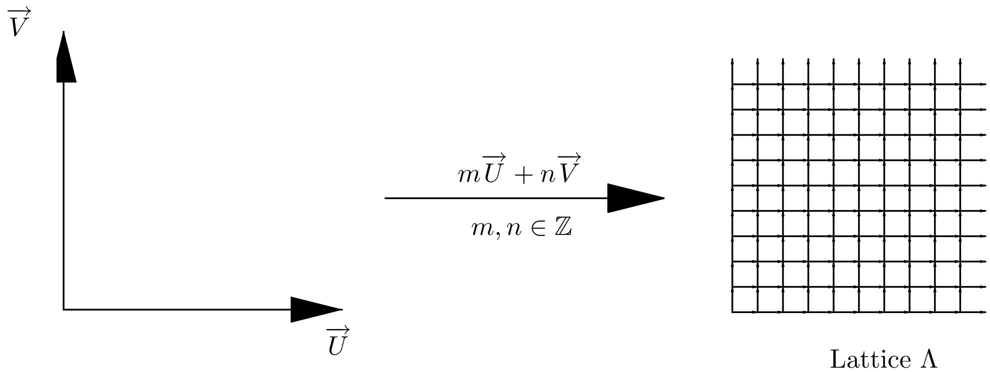

Fig. 1. Elementary square lattice.

This set presents a 4–fold rotational symmetry. Its symmetry space–group G is the semi–direct product of the translation–group of Λ by its rotation–group,

$$G = \Lambda \rtimes \{1, -1, \mathrm{e}^{\mathrm{i}\pi/2}, \mathrm{e}^{-\mathrm{i}\pi/2}\}\,. \tag{2}$$

The internal law is given by:

$$(\lambda, R)(\lambda', R') = (\lambda + R\lambda', RR')\,, \tag{3}$$

with $\lambda, \lambda' \in \Lambda$ and $R, R' \in \{1, -1, \mathrm{e}^{\mathrm{i}\pi/2}, \mathrm{e}^{-\mathrm{i}\pi/2}\}$.

Aperiodicity of quasicrystals implies the absence of such space–group structure based on the integers. On the other hand, experimentally observed quasicrystals show self-similarity (*e.g.* in their diffraction pattern). Those observed self-similarity factors are the quadratic Pisot–Vijayaraghavan (PV) units:

$$\beta = \tau = \frac{1+\sqrt{5}}{2}\,, \quad \beta = \delta = 1+\sqrt{2}\,, \quad \beta = \theta = 2+\sqrt{3}\,. \tag{4}$$

Each such β determines a discrete set of the line, $\mathbb{Z}_\beta$, the set of "beta–integers", aimed to play the role of integers. The first tau–integers around the origin are displayed in Fig. 2.

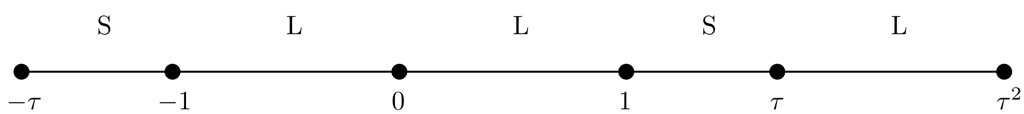

Fig. 2. First elements of $\mathbb{Z}_\tau$ (tau–integers) around the origin and associated tiling.

Beta–lattices are precisely aimed to replace lattices in the context of quasicrystals. They are based on beta–integers, like lattices are based on integers:

$$\Gamma = \sum_{i=1}^{d} \mathbb{Z}_\beta \mathbf{e}_i\,, \tag{5}$$

with $(\mathbf{e}_i)$ a base of $\mathbb{R}^d$.

In Fig. 3 we show the tau–lattice $\Gamma_1(\tau) = \mathbb{Z}_\tau + e^{i\pi/5}\mathbb{Z}_\tau$ in $\mathbb{R}^2$.

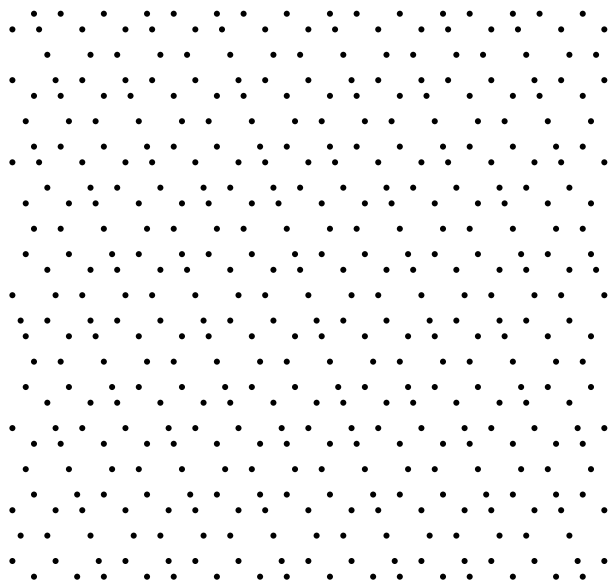

Fig. 3. τ–lattice $\Gamma_1(\tau) = \mathbb{Z}_\tau + e^{i\pi/5}\mathbb{Z}_\tau$ in $\mathbb{R}^2$.

Beta–lattices are eligible frames in which one could think of the properties of quasiperiodic point–sets and tilings, thus generalizing the notion of lattice in periodic cases.

As a matter of fact, it has become like a paradigm that geometrical supports of quasi–crystalline structures should be Delaunay sets obtained through "Cut-and-Projection" from higher–dimensional lattices. We show in Fig. 4 a "Cut-and-Project" 2D decagonal set and its embedding into the tau-lattice of Fig. 3 is shown in Fig. 5.

Therefore, within the context of aperiodic Crystallography it is natural to concede a place to what we can call *beta–periodic Crystallography.* Mimicking periodic Crystallography, we propose the following definitions.

– A beta–crystallographic group in $\mathbb{R}^d$ is a discrete group of beta–isometries (to be defined!) whose maximal translation subgroup is isomorphic to $(\mathbb{Z}_\beta)^d$.

– A beta–lattice or beta–periodic crystal is the orbit under the action of a beta–crystallographic group of a finite number of points of $\mathbb{R}^d$.

Of course, one could ask whether the domain of application of those definitions is empty or not. The content of this article, mainly based on the recent publication [2], yields specific examples illustrating this new concept of *beta–crystallographic group.*

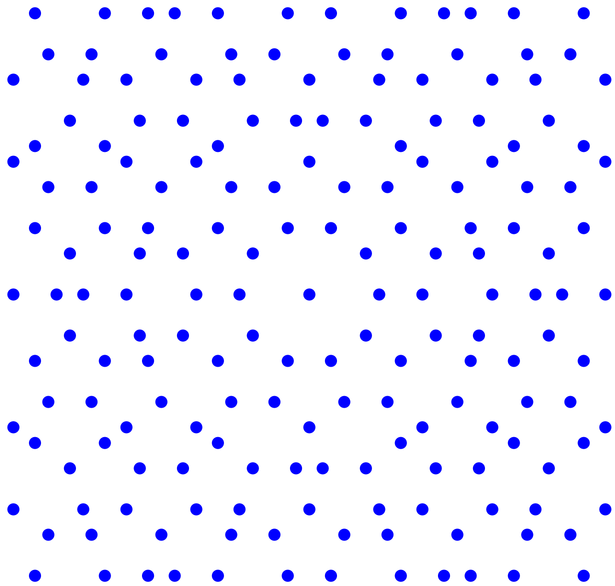

Fig. 4. A decagonal Cut-and-Project set

2 Beta−integers in place of integers

Let us define now in a precise way the beta–integers and give their fundamental arithmetic properties. More details and proofs are found in [3] and [4].

2.1 Counting with irrational basis

We first start with the notion of beta–expansions of real numbers [5, 6].

- Let $\beta > 1$.
- For a real number $x \geq 0$ there exists $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$. Let $x_k = \lfloor x/\beta^k \rfloor$ (integer part) and $r_k = \{x/\beta^k\}$ (fractional part).
- For $i < k$, put $x_i = \lfloor \beta r_{i+1} \rfloor$, and $r_i = \{\beta r_{i+1}\}$.
- The *beta–expansion* of a real number $x \geq 0$ then reads as:

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots + x_1\beta + x_0 + \frac{x_{-1}}{\beta} + \frac{x_{-2}}{\beta^2} \cdots \equiv$$
$$\equiv x_k x_{k-1} \cdots x_1 x_0 \cdot x_{-1} x_{-2} \cdots . \qquad (6)$$

- The digits x_i obtained by this *greedy* algorithm are integers from the set $A = \{0, \ldots, \lceil\beta\rceil - 1\}$ ($\lceil\beta\rceil$: smallest integer larger than β).

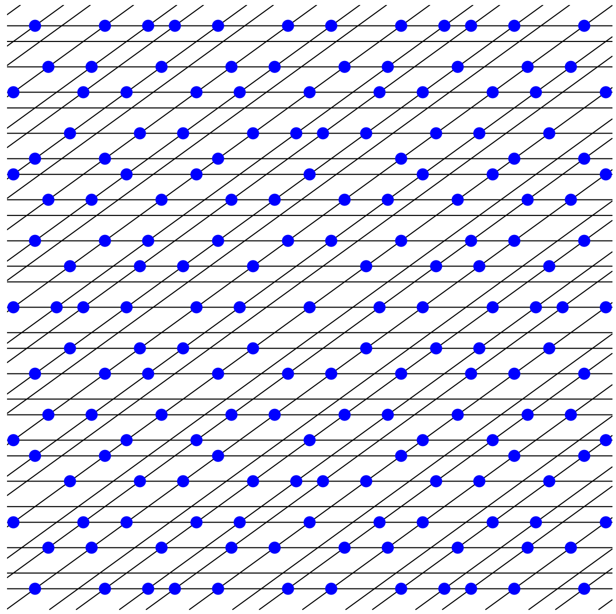

Fig. 5. Embedding of a Cut-and-Project set into the τ–lattice of Fig. 3.

Within this context, the set $\mathbb{Z}_\beta$ of beta–integers is made up of all real numbers whose beta–expansions are polynomial,

$$\begin{aligned} \mathbb{Z}_\beta &= \{x \in \mathbb{R} \mid |x| = x_k \cdots x_0\} = \\ &= \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+)\,. \end{aligned} \tag{7}$$

Set $\mathbb{Z}_\beta$ is self-similar and symmetrical with respect to the origin:

$$\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta\,, \quad \mathbb{Z}_\beta = -\mathbb{Z}_\beta\,. \tag{8}$$

If β is a PV number then $\mathbb{Z}_\beta$ is a Meyer set [3]. This means that there exists a finite set F such that $\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$. *This set F has to be characterized in order to see to what extent beta–integers differ from ordinary integers with respect to additive and multiplicative structures. This problem is solved for all quadratic Pisot–Vijayaraghavan (PV) units and for a few higher–degree cases* [3].

The quadratic PV units fall into the two following classes:

- Case 1: β is solution of $X^2 = aX + 1$, $a \geq 1$.

  * Define the 2–letter substitution σ_β by

$$\sigma_\beta : \begin{cases} L \mapsto L^a S\,, \\ S \mapsto L\,. \end{cases}$$

  * The fixed point of the substitution, denoted by $\sigma_\beta^\infty(L)$, is associated with a tiling of the positive real line, made with two tiles L and S, with respective lengths $\ell(L) = 1$, $\ell(S) = \beta - a = 1/\beta$.
  * The nodes of this tiling are the positive beta–integers.

- Case 2: β is solution of $X^2 = aX - 1$, $a \geq 3$.
  * Define the substitution σ_β by

$$\sigma_\beta : \begin{cases} L \mapsto L^{a-1}S\,, \\ S \mapsto L^{a-2}S\,. \end{cases}$$

  * The fixed point of the substitution is denoted by $\sigma_\beta^\infty(L)$ and is the tiling of the positive real line, made with two tiles L and S with respective lengths $\ell(L) = 1$, $\ell(S) = \beta - (a-1) = 1 - 1/\beta$.
  * The nodes of this tiling are the positive beta–integers.

The additive and multiplicative properties of beta–integers with β a PV are then given by:

- In Case 1 we have

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \left\{0, \pm\left(1 - \frac{1}{\beta}\right)\right\} \subset \mathbb{Z}_\beta/\beta^2, \tag{9}$$

$$\mathbb{Z}_\beta \times \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \left\{0, \pm\frac{1}{\beta}, \ldots, \pm\frac{a}{\beta}\right\} \subset \mathbb{Z}_\beta/\beta^2. \tag{10}$$

  For instance, for $\beta = \tau$, $1 + 1 = 2 = \tau + (1 - 1/\tau)$, and $(\tau^2 + 1)(\tau^2 + 1) = \tau^5 + \tau^2 - 1/\tau$.

- In Case 2 we have

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \left\{0, \pm\frac{1}{\beta}\right\} \equiv \widetilde{\mathbb{Z}}_\beta\,, \tag{11}$$

$$\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta^+/\beta, \tag{12}$$

$$\mathbb{Z}_\beta \times \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \left\{0, \pm\frac{1}{\beta}, \ldots, \pm\frac{a-1}{\beta}\right\} \subset \mathbb{Z}_\beta/\beta\,. \tag{13}$$

  ($\widetilde{\mathbb{Z}}_\beta$: set of "decorated" beta–integers)

  For instance, for $\beta = \theta$, $2 + 2 = \theta + 1/\theta = 2 \times 2$.

2.2 Beta−integers as an additive group [3, 4]

Let b_m and b_n be the m^{th} and n^{th} beta–integers. *Beta–addition* is the internal additive law on the set of beta–integers

$$b_m \oplus b_n = b_{m+n}\,. \tag{14}$$

We then list some remarkable consequences of this definition.

- $\mathbb{Z}_\beta$ is an abelian group for $\oplus$.
- Beta–addition is **compatible** with addition if β is a quadratic PV unit: for all $(m, n) \in \mathbb{Z}^2$, $b_m + b_n \in \mathbb{Z}_\beta$ implies $b_m + b_n = b_m \oplus b_n$.
- beta–addition has the following *minimal distortion property* with respect to addition: for all $(b_m, b_n) \in \mathbb{Z}_\beta^2$ with β a quadratic PV unit,

$$b_m + b_n - (b_m \oplus b_n) \in \begin{cases} \left\{0, \pm\left(1 - \dfrac{1}{\beta}\right)\right\} & \text{in Case 1,} \\ \left\{0, \pm\dfrac{1}{\beta}\right\} & \text{in Case 2}\,. \end{cases} \qquad (15)$$

- For instance, if $\beta = \tau$, then $1 \oplus 1 = \tau$ and $2 - \tau = 1 - \tau^{-1}$, and if $\beta = \theta$, then $2 \oplus 2 = \theta$ and $4 - \theta = \theta^{-1}$.

2.3 Quasi−multiplication on beta−integers [4]

We could try to play the same game with multiplication by defining

$$b_m \text{“}\times\text{”} b_n \stackrel{\text{def}}{=} b_{mn}\,, \qquad (16)$$

for all $(b_m, b_n) \in \mathbb{Z}_\beta^2$.

Actually we follow the wrong way in choosing (16) because *it is not* compatible with multiplication in $\mathbb{R}$. For instance, for $\beta = \tau$, $b_2 \times b_2 = \tau \times \tau = \tau^2 = b_3 \neq b_4$.

So we define the *quasi–multiplication*

$$b_m \otimes b_n = \begin{cases} b_{(mn - a\rho_S(m)\rho_S(n))} & \text{in Case 1,} \\ b_{(mn - \rho_S(m)\rho_S(n))} & \text{in Case 2,} \end{cases}$$

where, for $n \geq 0$, $\rho_S(n)$ denotes the number of tiles S between $b_0 = 0$ and b_n.

$$\begin{aligned} \rho_S(n) &= \frac{1}{1 - 1/\beta}(n - b_n)\,, \quad && \text{Case 1}\,, \\ \rho_S(n) &= \beta(n - b_n)\,, && \text{Case 2} \end{aligned}$$

for $n < 0$, $\rho_S(n) = -\rho_S(-n)$.

Let us list the properties of this quasi–multiplication

- Quasi–multiplication is **compatible** with multiplication of real numbers if β is a quadratic PV unit.
- Quasi–multiplication has minimal distortion property with respect to multiplication: for all $(b_m, b_n) \in \mathbb{Z}_\beta^2$ with β quadratic PV unit,

$$b_m b_n - (b_m \otimes b_n) \in \begin{cases} \left\{(0, \pm 1, \ldots, \pm a)\left(1 - \dfrac{1}{\beta}\right)\right\}, & \text{Case 1,} \\ \left\{(0, 1, \ldots, a-1)\dfrac{\operatorname{sgn}(b_m b_n)}{\beta}\right\}, & \text{Case 2.} \end{cases}$$

3 Beta−lattices in the plane

It is well known that the condition $2\cos(2\pi/N) \in \mathbb{Z}$, *i.e.* $N = 1, 2, 3, 4$ and 6, characterizes N–fold Bravais lattices in $\mathbb{R}^2$ (and in $\mathbb{R}^3$). Now, what can we do when N is quasicrystallographic *i.e.* $N = 5$, 10, 8 and 12, respectively associated with one of the cyclotomic Pisot units $\tau = 2\cos(2\pi/10)$, $\delta = 1 + 2\cos(2\pi/8)$ and $\theta = 2 + 2\cos(2\pi/12)$? Possible answers are provided by beta–lattices in the plane. We recall that they are point sets of the form

$$\Gamma_q(\beta) = \mathbb{Z}_\beta + \mathbb{Z}_\beta \zeta^q \,, \tag{17}$$

with $\zeta = \mathrm{e}^{\mathrm{i}2\pi/N}$, for $1 \leq q \leq N-1$. Examples of beta–lattices for $\beta = \tau$, δ, and θ are shown in Figs. 6, 7, and 8. Note the following important features.

- They are lattices for the law $\oplus$: $\Gamma_q(\beta) \oplus \Gamma_q(\beta) = \Gamma_q(\beta)$.
- They are self-similar: $\beta\Gamma_q(\beta) \subset \Gamma_q(\beta)$.
- They satisfy a more general "quasi" self-similarity: $\mathbb{Z}_\beta \otimes \Gamma \subset \Gamma$.
- However, they are not rotationally invariant.
- A large class of interesting aperiodic sets can be embedded in these beta–lattices $\Gamma_q(\beta)$ or in some "decorated" version of them.

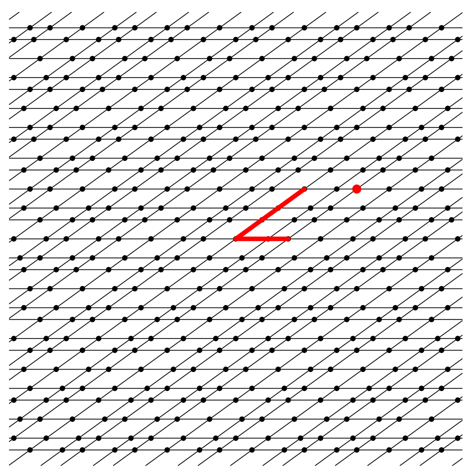

Fig. 6. Tau–lattice in the plane: $\Gamma_1(\tau) = \mathbb{Z}_\tau + \mathbb{Z}_\tau\zeta$, $\zeta = \mathrm{e}^{\mathrm{i}\pi/5}$. The particular point $z_{2,3} = b_2 + b_3\zeta \equiv (2,3)$ is indicated in the figure

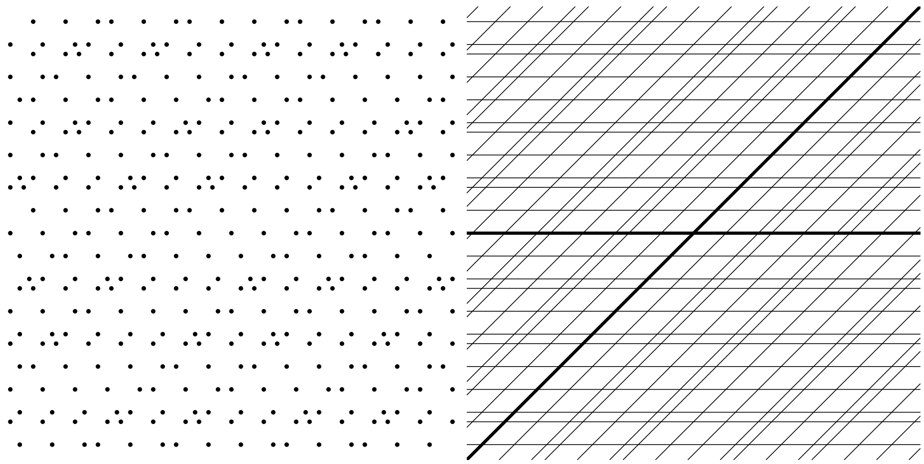

Fig. 7. The δ–lattice $\Gamma_1(\delta)$ with points, and its trivial tiling obtained by joining points along the horizontal axis, and along the direction defined by ζ.

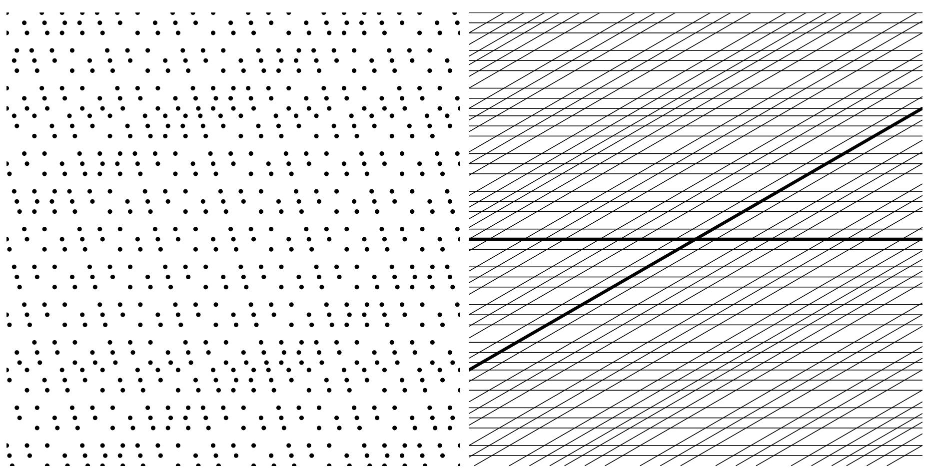

Fig. 8. The decorated θ–lattice $\widetilde{\Gamma}_1(\theta)$ with points, and its trivial tiling obtained by joining points along the horizontal axis, and along the direction defined by ζ.

4 Rotational and translational properties of the beta−lattices $\boldsymbol{\Gamma_1(\beta)}$

4.1 Rotational properties

Beta–lattices are not rotationally invariant. The action of rotations on $\Gamma_1(\beta)$ (the rotational properties of $\Gamma_q(\beta)$ can always be reexpressed in terms of the rotational properties of $\Gamma_1(\beta)$).

– When $\beta = \tau = \zeta + \bar{\zeta}$, $\zeta = \mathrm{e}^{2\pi \mathrm{i}/10}$,

$$\zeta^q \Gamma_1(\tau) \subset \Gamma_1(\tau) + \Big(\big\{0, \pm(1 - \tfrac{1}{\tau})\big\} + \big\{0, \pm(1 - \tfrac{1}{\tau})\big\}\zeta \Big) \subset \subset \frac{\Gamma_1(\tau)}{\tau^2} \,.$$

– When $\beta = \delta = \zeta + \bar{\zeta} + 1$, $\zeta = \mathrm{e}^{2\pi \mathrm{i}/8}$,

$$\zeta^q \Gamma_1(\delta) \subset \Gamma_1(\delta) + \Big(\big\{0, \pm(1 - \tfrac{1}{\delta}), \pm 2(1 - \tfrac{1}{\delta})\big\} + + \big\{0, \pm(1 - \tfrac{1}{\delta}), \pm 2(1 - \tfrac{1}{\delta})\big\}\zeta \Big) \subset \frac{\Gamma_1(\delta)}{\delta^3} \,.$$

– When $\beta = \theta = \zeta + \bar{\zeta} + 2$, $\zeta = \mathrm{e}^{2\pi \mathrm{i}/12}$,

$$\zeta^q \Gamma_1(\theta) \subset \Gamma_1(\theta) + \Big(\big\{0, \pm\frac{1}{\theta}, \pm\tfrac{2}{\theta}\big\} + \big\{0, \pm\tfrac{1}{\theta}, \pm\tfrac{2}{\theta}\big\}\zeta \Big) \subset \subset \tilde{\tilde{\Gamma}}_1(\theta) \equiv \tilde{\tilde{\mathbb{Z}}}_\theta + \tilde{\tilde{\mathbb{Z}}}_\theta \zeta \,,$$

where $\tilde{\tilde{\mathbb{Z}}}_\theta = \mathbb{Z}_\theta + \big\{0, \pm 1/\theta, \pm 2/\theta\big\}$.

4.2 Translational properties

– In Case 1:

$$\Gamma_q(\beta) + \Gamma_q(\beta) \subset \Gamma_q(\beta)/\beta^2 \,,$$

– In Case 2:

$$\Gamma_q(\beta) + \Gamma_q(\beta) \subset \tilde{\Gamma}_q(\beta) \,.$$

5 A plane−group for beta−lattices

5.1 A point group first

Since beta–lattices of the type $\Gamma_q(\beta)$ are not rotationally either translationally invariant, we shall enforce invariance by replacing the usual additive and multiplicative laws by the beta–addition and the quasi–multiplication. Note that since the quasi–multiplication is not distributive with respect to beta–addition, we find several candidates for internal rotational operators on $\Gamma_1(\beta)$. The choice for the beta–rotations presented here is driven by compatibility property. Other internal rotational operator are not compatible with Euclidean rotations**!**

Before stating the main result concerning the existence of a symmetry point group, let us describe in detail those necessary “modified” rotations.

– When $\beta = \tau$, the following 10 operators r_q, $q = 0, 1, \ldots, 9$, leave $\Gamma_1(\tau)$ invariant:

$$r_q \odot (b_m + b_n \zeta) = \eta_q b_m \ominus \nu_q b_n + \big(\nu_q b_m \oplus (\eta_q + \tau \nu_q) b_n\big)\zeta \,. \qquad (18)$$

where

$$\begin{array}{rccccc} q & = & 0 & 1 & 2 & 3 & 4 \\ (\eta_q, \nu_q) & = & (1,0) & (0,1) & (-1,\tau) & (-\tau,\tau) & (-\tau,1) \\ \eta_q + \nu_q \tau & = & 1 & \tau & \tau & 1 & 0 \end{array}$$

together with $(\eta_{q+5}, \nu_{q+5}) = (-\eta_q, -\nu_q)$.

– When $\beta = \delta$, the following operators leave $\Gamma_1(\delta)$ invariant:

$$\begin{aligned} r_1 \odot (b_m + b_n\zeta) &= -b_n + b_{m+2n-2\rho_S(n)}\zeta\,, \\ r_2 \odot (b_m + b_n\zeta) &= -b_{m+2n-2\rho_S(n)} + b_{2m+n-2\rho_S(m)}\zeta\,, \\ r_3 \odot (b_m + b_n\zeta) &= -b_{2m+n-2\rho_S(m)} + b_m\zeta\,. \end{aligned}$$

– When $\beta = \theta$, the following operators leave $\Gamma_1(\theta)$ invariant:

$$\begin{aligned} r_1 \odot (b_m + b_n\zeta) &= -b_n + b_{m+2n-\rho_S(n)}\zeta\,, \\ r_2 \odot (b_m + b_n\zeta) &= -b_{m+2n-\rho_S(n)} + b_{2m+2n-\rho_S(m)}\zeta\,, \\ r_3 \odot (b_m + b_n\zeta) &= -b_{2m+2n-\rho_S(m)} + b_{2m+2n-\rho_S(n)}\zeta\,, \\ r_4 \odot (b_m + b_n\zeta) &= -b_{2m+2n-\rho_S(m)} + b_{2m+n-\rho_S(n)}\zeta\,, \\ r_5 \odot (b_m + b_n\zeta) &= -b_{n+2m-\rho_S(m)} + b_m\zeta\,. \end{aligned}$$

For $\beta = \tau$, δ or θ, let the composition rule of these operators on $\Gamma_1(\beta)$ be defined by $(rr')\odot z = r\odot(r'\odot z)$, and denote by Id the identity and by ι the space inversion $\iota \odot z = -z$. Then:

– the composition rule $(r, r') \to rr'$ is associative and the following identities hold: $r_0 = \mathrm{Id}$ and $r_{q+N/2} = \iota r_q = r_q \iota$ for $q = 0, 1, \ldots, N/2-1$, where N is the symmetry order of β,

– those *beta–rotations* are compatible with rotations when β assumes one of the specified values τ, δ and θ,

– for $\beta = \tau$, δ and θ and for $N = 10$, 8 and 12 respectively, let $\Re_N = \Re_N(\beta)$ denote the semi–group freely generated by all r_q, $q \in \{0, 1, \ldots, N-1\}$. Among all beta–rotations, only $r_0, r_1, r_{N/2-1}, r_{N/2+1}, r_{N-1}, \iota$ have their inverse in $\Re_N$.

All these results lead to the following statement concerning the existence of point group for the considered beta–lattices.

Theorem 1 *[2] For $\beta = \tau$, δ and θ, the group $\mathcal{R}_N = \mathcal{R}_N(\beta)$, freely generated by the four element set*

$$\{r_0, \iota, r_1, r_{N/2-1}\}\,,$$

is a symmetry group for the beta–lattice $\Gamma_1(\beta)$. It is called the symmetry point–group *of $\Gamma_1(\beta)$.*

5.2 Eventually a plane group for beta−lattices $\mathbf{\Gamma_1(\beta)}$

Combining translational properties of beta–lattices with the beta–rotations described in the previous subsection allow us to enunciate our central result.

Theorem 2 *[2] For $\beta = \tau$, δ and θ, and for $N = 10$, 8 and 12 respectively, the group $\mathcal{S}_N = \mathcal{S}_N(\beta)$ freely generated by the five-element set $\{r_0, \iota, r_1, r_{N/2-1}, t_1\}$, with $t_1(z) = 1 \oplus z$, is a symmetry group for the beta–lattice $\Gamma_1(\beta)$. This group is the semi–direct product of $\Gamma_1(\beta)$ and $\mathcal{R}_N$*

$$\mathcal{S}_n = \Gamma_1(\beta) \rtimes \mathcal{R}_N \,,$$

with the composition rule

$$(b, R)(b', R') = (b \oplus (R \odot b'), RR') \,.$$

In the present context, $\mathcal{S}_N$ is called the symmetry plane–group *of* $\Gamma_1(\beta)$.

The action of an element of $\mathcal{S}_N$ on $\Gamma_1(\beta)$ is thus defined as

$$(b, R) \centerdot z = b \oplus (R \odot z) = t_b(R \odot z) \in \Gamma_1(\beta) \,.$$

5.3 Tile transformations using internal operations on $\mathbf{\Gamma_1(\tau)}$

In order to illustrate the beta–rotations, we consider again a tiling associated to the simplest beta–lattice, namely $\Gamma_1(\tau)$. This tiling is shown in Fig. 9. We display in Fig. 10 the (deforming) action of the "beta–rotation" r_1 on the four different types of tiles appearing in Fig. 9.

6 Asymptotic properties

Let us end this article with some considerations on the behavior of beta–lattices at large distances.

Let β be a quadratic PV unit number. Then the following asymptotic behaviour of beta–integers holds true

$$b_n \underset{|n|\to\infty}{\approx} \gamma n \,,$$
$$b_m \otimes b_n \underset{|m|,|n|\to\infty}{\approx} \gamma^2 mn \,.$$

where

$$\gamma = \begin{cases} 1 - \dfrac{1}{a}\left(1 - \dfrac{1}{\beta}\right)^2 = \dfrac{(a+2)\beta - a^2 - a - 2}{a} & \text{(Case 1)}\,, \\ 1 - \dfrac{1}{\beta^2} = a(\beta - a) + 2 & \text{(Case 2)}\,, \end{cases}$$

Hence, the multiplication $\otimes$ is *asymptotically* associative and distributive with respect to the addition $\oplus$. In this sense we can say that $\mathbb{Z}_\beta$ is asymptotically a

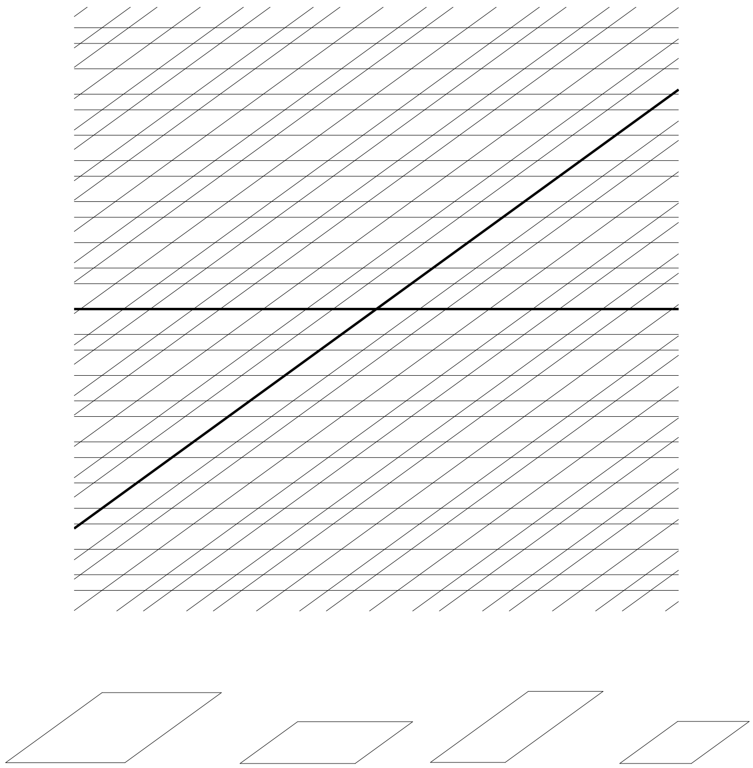

Fig. 9. The trivial tiling associated to $\Gamma_1(\tau)$ and its four tiles. From left to right: LL, LS, SL, SS.

ring:

$$b_m \otimes (b_n \oplus b_p) - (b_m \otimes b_n) \oplus (b_m \otimes b_p) \approx 0\,,$$
$$b_m \otimes (b_n \otimes b_p) - (b_m \otimes b_n) \otimes b_p \approx 0$$

for $|m|, |n|, |p|, |m \pm n|, |m \pm p| \to \infty$.

Consequently we compute the asymptotic behavior of rotational internal laws of beta–lattices, as defined in the studied cases.

• When $\beta = \tau$, we have for invertible operators

$$r_1 \odot (b_m + b_n\zeta) \underset{|m|,|n|\to\infty}{\approx} \gamma(-n + (m + \tau n)\zeta)\,,$$
$$r_4 \odot (b_m + b_n\zeta) \underset{|m|,|n|\to\infty}{\approx} \gamma(-\tau m - n - m\zeta)\,.$$

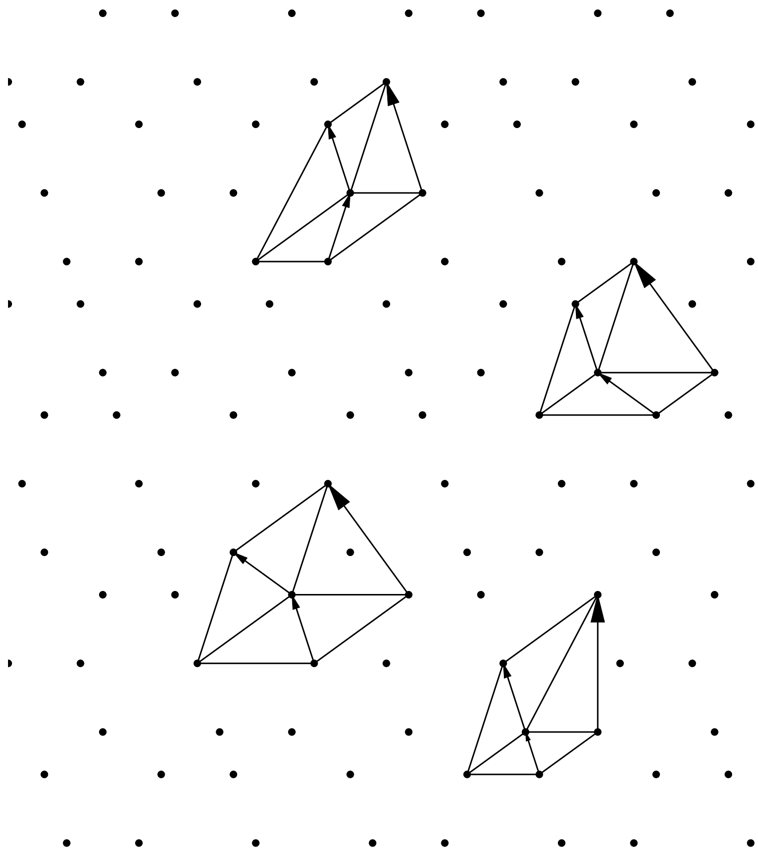

Fig. 10. Rotation operator r_1 applied to elementary tiles of $\Gamma_1(\tau)$: tiles are deformed in order for the vertices to remain in $\Gamma_1(\tau)$. Arrows indicate the vertices of the new tile in which are mapped the vertices of the original tile.

• When $\beta = \delta$, we have for invertible operators

$$r_1 \odot (b_m + b_n\zeta) \underset{|m|,|n|\to\infty}{\approx} \gamma(-n + (m + (\delta - 1)n)\zeta)\,,$$
$$r_3 \odot (b_m + b_n\zeta) \underset{|m|,|n|\to\infty}{\approx} \gamma(-(\delta - 1)m - n + m\zeta)\,.$$

• When $\beta = \theta$, we have for invertible operators

$$r_1 \odot (b_m + b_n\zeta) \underset{|m|,|n|\to\infty}{\approx} \gamma(-n + (m + (\theta - 2)n)\zeta)\,,$$
$$r_5 \odot (b_m + b_n\zeta) \underset{|m|,|n|\to\infty}{\approx} \gamma(-(\theta - 2)m - n + m\zeta)\,.$$

At this point one should be aware that these asymptotic beta–rotations are equivalent to rotations for large $|m|$ and $|n|$, and an easy computation shows that for

$z_{m,n} \in \Gamma_1(\beta)$

$$\zeta z_{m,n} - r_1 \odot z_{m,n} \underset{|m|,|n|\to\infty}{\approx} 0\,,$$
$$\zeta^{N/2-1} z_{m,n} - r_{N/2-1} \odot z_{m,n} \underset{|m|,|n|\to\infty}{\approx} 0\,,$$

with $N = 10$, 8 and 12.

References

[1] International Union of Crystallography: Acta Cryst. A **48** (1992) 922.

[2] A. Elkharrat, Ch. Frougny, J.P. Gazeau and J.L. Verger-Gaugry: Theor. Comp. Sci. **319** (2004) 281.

[3] C. Burdík, Ch. Frougny, J.P. Gazeau and R. Krejcar: J. of Physics A: Math. Gen. **31** (1998) 6449.

[4] Ch. Frougny, J.P. Gazeau and R. Krejcar: Theor. Comp. Sci. **303** (2003) 491.

[5] A. Rényi: Acta Math. Acad. Sci. Hungar. **8** (1957) 477.

[6] W. Parry: Acta Math. Acad. Sci. Hungar. **11** (1960) 401.