Non-Hermitian Hamiltonians and supersymmetric quantum mechanics *)

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We review non-Hermitian Hamiltonians following Mostafazadeh, while expanding on the underlying mathematical details. To conclude, we shortly summarize pseudo–supersymmetric quantum mechanics.

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1 Introduction

In standard quantum mechanics, the Hamiltonian or observable of energy is assumed to be given by a **self-adjoint** operator i.e. a Hilbert space operator H such that H and $H^\dagger$ act in the same way and such that the domains of definition of H and $H^\dagger$ coincide. Somewhat more vaguely, one can say that an observable is a *Hermitian* operator whose (generalized) eigenvectors define a (generalized) basis of Hilbert space, e.g. see [1]. The spectrum of a self-adjoint operator is real and describes the possible outcome for the measurement of the observable.

For the simplest quantum mechanical system, i.e. a 2–level system, the Hamiltonian is a self-adjoint operator on $\mathbf{C}^2$, i.e. a Hermitian (2×2)–matrix. Since there exist *non*-Hermitian matrices admitting real eigenvalues, such operators, as well as their generalization on infinite dimensional Hilbert space, are potential candidates for an attempt to extend standard quantum mechanics. In fact, various operators which act on the Hilbert space $L^2(\mathbf{R}^n)$ or on $\mathbf{C}^n$, which admit a real spectrum and are *not* Hermitian, have recently been put forward by Bender et al. [2] and have been largely discussed ever since. Many of these operators are $\mathcal{PT}$-invariant, where $\mathcal{P}$ and $\mathcal{T}$ denote the operations of parity and time reversal, respectively. On the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^n)$, the parity operator acts according to $(\mathcal{P}\varphi)(\vec{x}) = \varphi(-\vec{x})$ and on $\mathcal{H} = \mathbf{C}^2$ it acts as $\mathcal{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Time reversal acts as complex conjugation and thus represents an antilinear operator[1]). As a matter of fact, non-Hermitian Hamiltonians admit physical applications in such diverse areas as ionization optics, transitions in superconductors, dissipative quantum systems, nuclear potentials, quantum cosmology, population biology, ... — see the cited papers for precise references.

In our presentation of the described extension of quantum mechanics, we closely

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[1]) The adjoint of an antilinear operator $\mathcal{T}$ on $\mathcal{H}$ is defined by $\langle \varphi, \mathcal{T}\psi\rangle = \langle \mathcal{T}^\dagger\varphi, \psi\rangle^*$ for $\varphi, \psi \in \mathcal{H}$.

follow the approach of Mostafazadeh [3] – [8] dealing with 'pseudo–Hermitian' operators, while taking into account the mathematical and physical amendments put forward in references [9, 10]. The idea is to introduce a generalized adjoint operator defined in terms of some metric operator on the Hilbert space and to consider the operators which are Hermitian with respect to this generalized adjoint. The relationship with operators admitting an antilinear symmetry like $\mathcal{PT}$–symmetry is made in a second stage.

2 Pseudo−Hermitian operators

As in ordinary quantum mechanics, we consider a complex separable Hilbert space $\mathcal{H}$ with a scalar product $\langle \cdot, \cdot \rangle$ to which we refer as the *defining* scalar product of our Hilbert space. Let η be an operator on $\mathcal{H}$ which is defined on the entire space $\mathcal{H}$, which is bounded, Hermitian and invertible[2]). This operator is referred to as **metric** since it can be used to define the so-called **$\boldsymbol{\eta}$−scalar product** on $\mathcal{H}$:

$$\langle\!\langle \varphi, \psi \rangle\!\rangle_\eta \equiv \langle \varphi, \eta\psi \rangle \quad \text{for all} \ \ \varphi, \psi \in \mathcal{H} \, . \tag{1}$$

In general, this only represents a *pseudo–scalar product* since it is not necessarily positive–definite, but only non-degenerate: if $\langle\!\langle \varphi, \psi \rangle\!\rangle_\eta = 0$ for all $\varphi \in \mathcal{H}$, then $\psi = 0$ since η is invertible and thus $\ker \eta = \{0\}$. (In this context, it is worth recalling that the Minkowski metric operator appearing in special relativity, which is also denoted by η in general, represents a pseudo–scalar product on the real vector space $\mathbf{R}^4$.)

Two Hilbert space vectors φ and ψ satisfying $\langle\!\langle \varphi, \psi \rangle\!\rangle_\eta = 0$ are said to be η–*orthogonal*. In Dirac's terminology, the bra associated to the ket $|\psi\rangle \in \mathcal{H}$ with respect to the η–bracket (1) is given by $\langle \eta\psi|$.

The η–scalar product is *positive definite* if the operator η is *strictly positive* (i.e. $\langle \varphi, \eta\varphi \rangle > 0$ for all $\varphi \neq 0$ [12]) and thus of the form

$$\eta = \mathcal{O}^\dagger \mathcal{O} \, , \tag{2}$$

where $\mathcal{O}$ is an everywhere defined, bounded and invertible operator on $\mathcal{H}$. The simplest example is given by $\eta = \mathbb{1}$, in which case the η–scalar product reduces to the defining scalar product $\langle \cdot, \cdot \rangle$ of $\mathcal{H}$.

In these notes, we have in mind Hamiltonian operators and therefore we generically denote Hilbert space operators by H. The **$\boldsymbol{\eta}$−pseudo−adjoint of $\boldsymbol{H}$** (or pseudo-adjoint of H with respect to η) is defined by

$$H^\sharp = \eta^{-1} H^\dagger \eta \, . \tag{3}$$

A short calculation shows that $H^\sharp$ can be viewed as the adjoint of H with respect to the η–scalar product:

$$\langle\!\langle \varphi, H\psi \rangle\!\rangle_\eta = \langle\!\langle H^\sharp\varphi, \psi \rangle\!\rangle_\eta \qquad \text{for} \ \ \varphi, \psi \in \mathcal{H} \, .$$

[2]) Here and in the following, **invertible** means bijective, i.e. one-to-one and onto. Henceforth, the operator η^{-1} has the same properties as η. Concerning these properties, it is useful to recall the fundamental theorem of Hellinger and Toeplitz which states that an everywhere defined, Hermitian operator on Hilbert space is bounded [11].

We note that the metrics η and $\eta' = \lambda\eta$ (with $\lambda \in \mathbf{R} - \{0, 1\}$), which define different pseudo–scalar products, determine the same operation $\sharp$.

The operator H is said to be **$\boldsymbol{\eta}$−pseudo−Hermitian** (or pseudo–Hermitian with respect to η) if $H^\sharp = H$ (i.e. $H^\dagger = \eta H \eta^{-1}$ or $H^\dagger \eta = \eta H$, which means that the metric operator η intertwines between H and its adjoint). More generally, H is called **pseudo−Hermitian** if there exists a metric η on $\mathcal{H}$ with respect to which H is pseudo–Hermitian[3]. In particular, a **pseudo−Hermitian quantum mechanical system** $(\mathcal{H}, H)$ is given by a pseudo–Hermitian Hamiltonian H acting on the Hilbert space $\mathcal{H}$. In the case where η is strictly positive, a η–pseudo–Hermitian operator is said to be **$\boldsymbol{\eta}$−quasi−Hermitian** [10].

The operation $\sharp$ satisfies $(H^\sharp)^\sharp = H$ and shares other characteristic properties of the operation $\dagger$. Thus, the passage from standard to pseudo–Hermitian quantum mechanics consists of replacing $H^\dagger$ by $H^\sharp$, which leads to different spectral properties for the operators H that are invariant under the operations $\dagger$ and $\sharp$, respectively. In section 5, we will summarize the main results concerning the spectrum of pseudo–Hermitian and quasi–Hermitian operators. Here, we only make some related comments.

Remarks: (1) An operator which is pseudo–Hermitian with respect to $\eta = \mathbb{1}$ is Hermitian.

(2) If an operator H is pseudo–Hermitian with respect to the metric η, i.e. satisfies $H = \eta^{-1} H^\dagger \eta$, then it is also pseudo–Hermitian with respect to the metric

$$\eta' = S^\dagger \eta S \,, \tag{4}$$

where S is an everywhere defined, bounded and invertible operator commuting with H. From these properties of S, it follows that the metric η' is strictly positive *if and only if* η is strictly positive. Thus, there is a whole class of metrics associated with a given pseudo–Hermitian operator, all of which metrics are strictly positive if one of them is. Actually, one can consider more general changes of the metric which leave the pseudo–Hermicity condition for H invariant and which modify the positivity properties of η, see section 4. In that section, we will also see that, for a diagonalizable pseudo–Hermitian operator H, the eigenvectors of H and $H^\dagger$ can be used to obtain a simple explicit expression for a metric operator with respect to which H is pseudo–Hermitian.

3 Quasi−Hermitian operators on a finite dimensional space

Let us discuss the spectral theory of quasi–Hermitian operators while assuming that the Hilbert space $\mathcal{H}$ is of *finite* dimension so as to avoid technical complications. It is worthwhile recalling that the spectra of the operators H and $H^\dagger$ are mirror images of each other with respect to the real axis. More precisely, the eigenvalues

[3]) Here, we do not discuss the domains of definition of the involved operators — see reference [13] for some related considerations.

of $H^\dagger$ are the complex conjugate of the eigenvalues of H, with the same algebraic and geometric multiplicities [14].

According to the definition given above, an operator H is quasi–Hermitian if it is Hermitian with respect to a positive–definite η–scalar product: $\langle\!\langle \varphi, H\psi\rangle\!\rangle_\eta = \langle\!\langle H\varphi, \psi\rangle\!\rangle_\eta$ for all $\varphi, \psi \in \mathcal{H}$. Thus, these operators have spectral properties that are analogous to the standard Hermitian operators which satisfy $\langle \varphi, H\psi\rangle = \langle H\varphi, \psi\rangle$ for all $\varphi, \psi \in \mathcal{H}$. Indeed, by applying the standard arguments to the η–scalar product rather than the scalar product $\langle\cdot,\cdot\rangle$, one concludes that *the eigenvalues of a quasi–Hermitian operator H are real,* that the eigenvectors associated to different eigenvalues are mutually η–orthogonal and that the eigenvectors provide an η–orthonormal basis[4]):

$$\begin{aligned} H|\psi_{n,a}\rangle &= E_n\,|\psi_{n,a}\rangle\,, \\ \langle\!\langle \psi_{n,a}, \psi_{m,b}\rangle\!\rangle_\eta &= \delta_{nm}\delta_{ab}\,, \\ \sum_n \sum_{a=1}^{d_n} |\psi_{n,a}\rangle\langle \eta\psi_{n,a}| &= \mathbb{1} = \sum_n \sum_{a=1}^{d_n} |\eta\psi_{n,a}\rangle\langle \psi_{n,a}|\,. \end{aligned} \tag{5}$$

Here, n labels the eigenvalues E_n of H and $a \in \{1,\dots,d_n\}$ their degeneracy (d_n being the multiplicity of the eigenvalue E_n). The given decomposition of unity follows from the fact that the eigenvectors of an operator which is Hermitian with respect to some positive–definite scalar product form a complete system, hence any vector $|\psi\rangle \in \mathcal{H}$ can be expanded with respect to this system: $|\psi\rangle = \sum_n \sum_{a=1}^{d_n} c_{n,a}|\psi_{n,a}\rangle$. By multiplying this expansion to the left by $\eta\psi_{m,b}$, we get $c_{n,a} = \langle \eta\psi_{n,a}, \psi\rangle = \langle\!\langle \psi_{n,a}, \psi\rangle\!\rangle_\eta$ which is precisely the first resolution of identity. The second follows from the first one by writing $\mathbb{1} = \eta\mathbb{1}\eta^{-1}$ and using the Hermicity of η^{-1}. From $H = \mathbb{1}H\mathbb{1}$ and equations (5), we get the *spectral decomposition of H*:

$$H = \sum_n \sum_{a=1}^{d_n} E_n\,|\psi_{n,a}\rangle\langle \eta\psi_{n,a}|\,. \tag{6}$$

For $\eta \neq \mathbb{1}$, the basis $\{|\psi_{n,a}\rangle\}$ is not orthonormal (with respect to the defining scalar product $\langle\cdot,\cdot\rangle$). Equations (5) and (6) involve the vectors

$$|\phi_{n,a}\rangle \equiv \eta|\psi_{n,a}\rangle\,, \tag{7}$$

which also form a basis of $\mathcal{H}$ since the operator η is invertible. For $\eta \neq \mathbb{1}$, this basis is not orthonormal either. From $H^\dagger = \eta H\eta^{-1}$ it follows that $H^\dagger|\phi_{n,a}\rangle = E_n\,|\phi_{n,a}\rangle$ so that the spectral decompositions of H and $H^\dagger$ read as

$$H = \sum_n \sum_{a=1}^{d_n} E_n\,|\psi_{n,a}\rangle\langle \phi_{n,a}|\,, \quad H^\dagger = \sum_n \sum_{a=1}^{d_n} E_n\,|\phi_{n,a}\rangle\langle \psi_{n,a}|\,. \tag{8}$$

[4]) Here and in the following, it is convenient to use Dirac's bra and ket notation.

The expansion of H involves the projection operators $P_{n,a} \equiv |\psi_{n,a}\rangle\langle\phi_{n,a}|$ which are not Hermitian in general: $P^\dagger_{n,a} = |\phi_{n,a}\rangle\langle\psi_{n,a}|$. The η–orthonormality and η–completeness relations (5) now take the form

$$\begin{aligned} \langle \phi_{n,a}, \psi_{m,b}\rangle &= \delta_{nm}\delta_{ab}\,, \\ \sum_n \sum_{a=1}^{d_n} |\psi_{n,a}\rangle\langle\phi_{n,a}| &= \mathbb{1} = \sum_n \sum_{a=1}^{d_n} |\phi_{n,a}\rangle\langle\psi_{n,a}|\,, \end{aligned} \tag{9}$$

i.e. the eigenvectors of H and $H^\dagger$ form a **complete biorthogonal system** in $\mathcal{H}$ [15]. The decomposition of unity (9) means that any vector $|\psi\rangle \in \mathcal{H}$ can be decomposed either with respect to the basis $\{|\psi_{n,a}\rangle\}$ or with respect to the basis $\{|\phi_{n,a}\rangle\}$. We remark that biorthogonal systems play an important role in the theory of nonharmonic Fourier series and for wavelet expansions [15, 16, 17].

For the biorthogonal system constructed above, we have relation (7), i.e. the basis $\{|\psi_{n,a}\rangle\}$ and the basis $\{|\phi_{n,a}\rangle\}$ are related by the invertible Hermitian operator η. From this relationship, we can deduce an explicit expression for the metric η in terms of the eigenvectors of $H^\dagger$: by writing $\eta = \eta\mathbb{1}$ and using relations (9), we find

$$\eta = \sum_n \sum_{a=1}^{d_n} |\phi_{n,a}\rangle\langle\phi_{n,a}|\,. \tag{10}$$

This expression is referred to as the *canonical representation* of the (strictly positive) metric operator η.

To summarize: a η–quasi–Hermitian operator H is diagonalizable with *real* eigenvalues (see next subsection for a further discussion), i.e. its eigenvalues are real and the set of eigenvectors $\{|\psi_{n,a}\rangle, |\phi_{n,a}\rangle\}$ of H and $H^\dagger$,

$$H|\psi_{n,a}\rangle = E_n\,|\psi_{n,a}\rangle\,, \quad H^\dagger|\phi_{n,a}\rangle = E_n\,|\phi_{n,a}\rangle\,, \tag{11}$$

forms a complete biorthogonal system in $\mathcal{H}$ (for which the relation (7) holds due to the η–quasi–Hermiticity of H). Conversely, an operator H with the eigenvalue equations (11), that involve *real* eigenvalues E_n and eigenvectors which form a *complete biorthogonal system* in $\mathcal{H}$, is quasi–Hermitian with respect to the metric (10). Indeed, the operator η defined by (10) is Hermitian, strictly positive and invertible with inverse $\eta^{-1} = \sum_n \sum_{a=1}^{d_n} |\psi_{n,a}\rangle\langle\psi_{n,a}|$. Furthermore, equations (10) and (9) imply relation (7), while the spectral decompositions of H and $H^\dagger$ imply $\eta^{-1}H^\dagger\eta = H$, i.e. the η–quasi–Hermiticity of H. Thus, for an operator H on a complex Hilbert space of finite dimension, we have established the equivalence between *quasi–Hermitian* and *diagonalizable with real eigenvalues*. Another characterization of quasi-Hermitian operators will be presented in section 5.

4 Diagonalizable operators, metrics and symmetries

At the beginning of this section, we recall some facts concerning the diagonalization and spectral decomposition of non-Hermitian operators in Hilbert space without any reference to the notion of pseudo–Hermiticity. Thereafter, we provide an explicit expression for the metric for a given pseudo–Hermitian operator H and we explain how one can find antilinear symmetries for such an operator [3, 4, 5]. To start with, we avoid the mathematical technicalities by considering the case of a finite dimensional Hilbert space.

4.1 Finite dimensional case

Spectral decompositions:

For a non-Hermitian operator, the eigenvectors associated to different eigenvalues are not necessarily orthogonal to each other and in general the eigenvectors do not provide a basis. Yet, the eigenvectors associated to different eigenvalues are always linearly independent. We recall once more that the spectra of the operators H and $H^\dagger$ are related by complex conjugation.

Interestingly enough, the spectral theorem for Hermitian operators admits a generalization which holds for arbitrary operators: the *canonical* or *singular value decomposition* [18] of a **generic operator** H reads as

$$H = \sum_k \sigma_k |v_k\rangle\langle w_k| \, , \quad \text{where} \quad \begin{cases} H^\dagger H |w_k\rangle = \sigma_k^2 |w_k\rangle \, , \\ H H^\dagger |v_k\rangle = \sigma_k^2 |v_k\rangle \, . \end{cases} \tag{12}$$

Here, the σ_k are the *singular values* of H (i.e. the non-negative square roots of the eigenvalues of the positive operator $H^\dagger H$) and the expansion (12) of H involves two orthonormal basis' consisting, respectively, of the eigenvectors of $H^\dagger H$ and $HH^\dagger$. Since the canonical decomposition of the operator H does not directly involve its eigenvalues, it is of limited interest for the physical interpretation of quantum mechanics, though it may be exploited in the mathematical study of pseudo–Hermitian Hamiltonians within the framework of perturbation theory [19].

Hermitian operators belong to the class of **normal** operators, i.e. Hilbert space operators H_0 satisfying $[H_0, H_0^\dagger] = 0$. Although these operators are not Hermitian in general, they admit exactly the same spectral decomposition as Hermitian operators, but involving complex eigenvalues. A normal operator can also be characterized by the property that there exists a Hilbert space basis consisting of *orthonormal eigenvectors* of H_0. Equivalently, the operator H_0 admits the spectral decomposition

$$H_0 = \sum_n \sum_{a=1}^{d_n} E_n \, |n,a\rangle\langle n,a| \, , \tag{13}$$

where the labels n, a and d_n have the same meaning as in equation (5), where E_n is complex in general and where the set $\{|n,a\rangle\}$ represents an orthonormal basis consisting of eigenvectors of H_0. For obvious reasons, these operators are also referred to as **diagonal** operators.

Relation (13) implies $H_0^\dagger = \sum_n \sum_{a=1}^{d_n} E_n^* \, |n,a\rangle\langle n,a|$, henceforth a normal operator with real eigenvalues is Hermitian. Thus, non-Hermitian operators with real eigenvalues, i.e. the central objects of our investigation, necessarily belong to a larger class of operators than the normal ones.

For our purposes, the latter class has to be constrained by physical requirements. Indeed [3], in quantum mechanics, one is interested in operators H which are **diagonalizable**, i.e. there exists an invertible operator S and a diagonal operator H_0 such that $H = S^{-1}H_0S$ (or, equivalently, there exists a Hilbert space basis consisting of eigenvectors of H). The condition that H is diagonalizable is also equivalent to the fact that the eigenvectors of H and $H^\dagger$, as given by equations

$$H|\psi_{n,a}\rangle = E_n\,|\psi_{n,a}\rangle\,, \quad H^\dagger|\phi_{n,a}\rangle = E_n^*\,|\phi_{n,a}\rangle\,, \tag{14}$$

form a complete biorthogonal system in $\mathcal{H}$, i.e. satisfy the set of relations (9). In fact [4], let us assume that $H = S^{-1}H_0S$ with H_0 of the form (13) and let us define

$$|\psi_{n,a}\rangle := S^{-1}|n,a\rangle\,, \quad |\phi_{n,a}\rangle := S^\dagger|n,a\rangle\,.$$

Then, relations (14) and (9) hold. Conversely, let us assume that (14) and (9) are satisfied and let us define $S^{-1} := \sum_{n,a}|\psi_{n,a}\rangle\langle n,a|$ where $\{|n,a\rangle\}$ denotes an orthonormal basis. Then, $S = \sum_{n,a}|n,a\rangle\langle\phi_{n,a}|$ and $SHS^{-1} = H_0$ with H_0 of the form (13).

From equations (9) and (14), we get the *spectral decomposition of a diagonalizable operator H*:

$$H = \sum_n\sum_{a=1}^{d_n}E_n\,|\psi_{n,a}\rangle\langle\phi_{n,a}|\,, \quad H^\dagger = \sum_n\sum_{a=1}^{d_n}E_n^*\,|\phi_{n,a}\rangle\langle\psi_{n,a}|\,. \tag{15}$$

If H is normal, then $\{|\psi_{n,a}\rangle = |\phi_{n,a}\rangle\}$ represents an orthornormal basis of $\mathcal{H}$ and the expansion (15) of H reduces to the expansion (13). If H is Hermitian, the eigenvalues are, in addition, real.

Finally, let us recall that there exists a simple **criterion for the diagonalizability** of an operator H on a complex Hilbert space $\mathcal{H}$ of finite dimension. Since the field of complex numbers is algebraically complete, the characteristic polynomial of an operator H on $\mathcal{H}$ can always be decomposed into linear factors, henceforth a necessary and sufficient condition for the diagonalizability of H is that the algebraic and geometric multiplicities coincide for each of its eigenvalues.

In section 3, we have shown that quasi–Hermitian is equivalent to diagonalizable with real eigenvalues. We will see in section 5 that, for a diagonalizable Hamiltonian H, the property pseudo–Hermitian is equivalent to the property that the eigenvalues of H are real or that they represent complex conjugate pairs of non-real numbers.

A non-diagonalizable operator H on a complex Hilbert space $\mathcal{H}$ is **block−diagonalizable**, i.e. $\mathcal{H}$ admits a basis with respect to which the operator H is represented by a block–diagonal matrix involving Jordan blocks [20]. In this case, the spectral decomposition of H [8] involves higher order eigenvectors (corresponding to the Jordan blocks): by definition, an *eigenvector* or *root vector of order* $m \in \{1,2,\dots\}$ associated to the eigenvalue E_n is a nonzero vector $|\psi_n\rangle \in \mathcal{H}$ such that

$$(H - E_n\mathbb{1})^m|\psi_n\rangle = 0\,, \quad (H - E_n\mathbb{1})^{m-1}|\psi_n\rangle \neq 0\,.$$

Thus, for $m = 1$, $|\psi_n\rangle$ is an ordinary eigenvector. For $m > 1$, the geometric and algebraic multiplicities of E_n do not coincide. Block–diagonalizable Hamiltonians manifest themselves in quantum mechanics in the description of energy level–crossings — see section 5.

Pseudo−Hermitian operators and associated metrics:

Suppose the operator H is diagonalizable, i.e. there exists a biorthogonal basis of eigenvectors $\{|\psi_{n,a}\rangle, |\phi_{n,a}\rangle\}$. If H is, in addition, pseudo–Hermitian, then one can use the latter biorthogonal basis to give an explicit expression for a metric operator η with respect to which H is pseudo–Hermitian [6]. In fact, the metric can be brought into the *canonical form*

$$\eta = \sum_{n_0} \sum_{a=1}^{d_{n_0}} \sigma_{n_0,a} |\phi_{n_0,a}\rangle\langle\phi_{n_0,a}| + \sum_{\nu} \sum_{a=1}^{d_\nu} \big(|\phi_{\nu,a}\rangle\langle\phi_{-\nu,a}| + |\phi_{-\nu,a}\rangle\langle\phi_{\nu,a}| \big) \, . \qquad (16)$$

Here, the index n_0 labels real eigenvalues, the index $\pm\nu$ labels the non-real eigenvalues which come in complex conjugate pairs ($E_{-\nu} = E_\nu^*$) and $\sigma_{n_0,a} \in \{-1, +1\}$ represents a collection of signs. If the spectrum of H is real, only the first double sum is present in expression (16) and one recovers the expansion (10) up to the signs.

If the biorthogonal basis is changed by an invertible operator S, the metric transforms according to (4).

Antilinear operators and symmetries:

The following definitions are formulated in such a way that they also apply in the case of an infinite dimensional Hilbert space.

Suppose the Hamiltonian H admits an *antilinear symmetry,* i.e. there exists an antilinear, everywhere defined, bounded operator χ which commutes with H. By applying χ to the eigenvalue equation $H|\psi_{n,a}\rangle = E_n|\psi_{n,a}\rangle$, we conclude that

$$H(\chi|\psi_{n,a}\rangle) = E_n^* \, \chi|\psi_{n,a}\rangle \, .$$

Thus, the eigenvalues of H are real or they come in pairs of complex conjugate non-real numbers [2, 3].

Let us now consider an operator τ which has the same properties as a metric operator except that it is *anti-Hermitian* rather than Hermitian, i.e. it is antilinear and satisfies $\langle\varphi, \tau\psi\rangle = \langle\tau\varphi, \psi\rangle^*$ for all $\varphi, \psi \in \mathcal{H}$. The Hamiltonian is said to be **pseudo−anti−Hermitian** with respect to τ if $H^\dagger = \tau H \tau^{-1}$. One can prove that every diagonalizable Hamiltonian is pseudo–anti–Hermitian with respect to an operator τ which is unique up to basis transformations [4, 6]. Furthermore, a short calculation shows that, if the Hamiltonian H is pseudo–Hermitian with respect to the metric η and pseudo–anti–Hermitian with respect to τ, the operator $\chi \equiv \eta^{-1}\tau$ represents an antilinear symmetry of H (which, in addition, is anti–Hermitian and invertible). By combining the previous results, one concludes that *every diagonalizable pseudo–Hermitian Hamiltonian H admits an antilinear symmetry.* If $\{|\psi_{n,a}\rangle, |\phi_{n,a}\rangle\}$ represents a biorthogonal basis of eigenvectors associated to such an operator, a *canonical expression for the antilinear symmetry* can be given in terms of this basis [4, 5]:

$$\chi = \sum_{n_0} \sum_{a=1}^{d_{n_0}} \sigma_{n_0,a} \, |\psi_{n_0,a}\rangle \star \langle\phi_{n_0,a}| + \sum_{\nu} \sum_{a=1}^{d_\nu} \big(|\psi_{\nu,a}\rangle \star \langle\phi_{-\nu,a}| + |\psi_{-\nu,a}\rangle \star \langle\phi_{\nu,a}| \big) \, . \tag{17}$$

Here, the symbol $\star$ denotes the operator of complex conjugation: $\star\langle\phi_{n,a}|\psi\rangle = \langle\phi_{n,a}|\psi\rangle^*$ for all $|\psi\rangle \in \mathcal{H}$.

4.2 Infinite dimensional case

For operators on an infinite dimensional Hilbert space, there are two well known technical complications, e.g. see reference [1]. First, the *spectrum* of an operator is defined to be the set of all *spectral values* and this set contains the subset of all *eigenvalues* (associated to eigenvectors belonging to the Hilbert space) and the subset of all *generalized eigenvalues* (associated to generalized eigenvectors which do not belong to the Hilbert space), these subsets corresponding to the *discrete* and *continuous spectrum*, respectively[5]. Second, for an unbounded operator, one has to deal with its *domain of definition* whose choice reflects in general a specific physical situation. These technical complications imply that functional analysis is much more harder and richer than linear algebra. In the appendix, we address the question to which extent the results of the previous subsection can be generalized to an infinite dimensional Hilbert space. Here, we only note two things. First, the formulae involving eigenvalues and eigenvectors which have been given above for operators acting on a finite dimensional Hilbert space, can be generalized to an operator with a continuous and/or discrete spectrum by considering spectral values, generalized eigenvectors, integrals and/or infinite sums [21]. Second, we remark that, for a given Hamiltonian on $L^2(\mathbf{R}^n)$, it is in general not possible to determine explicitly the spectrum and (generalized) eigenfunctions and it is difficult to determine whether or not such a Hamiltonian is diagonalizable, i.e. establish the completeness of the system of eigenfunctions.

5 Spectral theory for diagonalizable pseudo−Hermitian operators

5.1 General results

Theorem 5.1 (Spectral characterization of pseudo−Hermitian operators) *Let H be diagonalizable. Then, the following statements are equivalent:*

(P1) *H is pseudo–Hermitian.*

(P2) *The spectral values of H are real or they come in complex conjugate pairs of non-real numbers with the same multiplicities.*

[5]) Actually, there exists a third subset, the so-called *residual spectrum* which is empty for self-adjoint operators, but not for generic operators [14].

(P3) *There exists an antilinear symmetry of H, i.e. an antilinear, everywhere defined, bounded and invertible operator which commutes with H.*

Theorem 5.2 (Spectral characterization of quasi−Hermitian operators) *The following statements are equivalent:*

(Q1) *H is quasi–Hermitian, i.e. pseudo–Hermitian with respect to a metric of the form $\eta = \mathcal{O}^\dagger \mathcal{O}$, see equation (2).*

(Q2) *H is diagonalizable and admits real spectral values.*

(Q3) *There exists a self-adjoint operator $\widehat{H}$ and a bounded invertible operator $\mathcal{O}$ such that*

$$H = \mathcal{O}^{-1} \widehat{H} \mathcal{O} . \tag{18}$$

(This operator is then quasi–Hermitian with respect to the metric $\eta = \mathcal{O}^\dagger \mathcal{O}$.)

Remarks:

(1) Since the operation $\mathcal{T}$ of time reversal (complex conjugation) in quantum mechanics is antilinear, every $\mathcal{PT}$–invariant, diagonalizable Hamiltonian is pseudo–Hermitian by virtue of the first theorem.

(2) Consider a continuous, pseudo–Hermiticity–preserving perturbation of a pseudo–Hermitian diagonalizable Hamiltonian. Under such a perturbation, a pair of non-degenerate complex conjugate eigenvalues may merge into a real eigenvalue. At such an energy level–crossing, the Hamiltonian generally looses its diagonalizability [8]. At this point, one therefore has to refer to the theory of pseudo-Hermitian block–diagonalizable operators for which a spectral characterization has also been given [8].

(3) The equivalence between (Q1) and (Q3) can readily be checked by writing out the expressions in terms of the operator $\mathcal{O}$. The equivalence between (Q1) and (Q2) was established in section 3. The results presented here concerning quasi–Hermitian operators are slightly stronger than those obtained by Mostafazadeh [3, 4, 6] in that we do not assume the operator H to be diagonalizable to start with (as one does in the first theorem).

(4) According to the second theorem, every diagonalizable Hamiltonian with real spectral values can be related to a self-adjoint Hamiltonian by a similarity transformation. In this sense, quasi–Hermitian Hamiltonians are equivalent to self-adjoint ones [7] — see next subsection for a simple example.

Let us now present some prototype examples for operators on the Hilbert space $\mathbf{C}^2$ (following reference [6]) and for operators on the Hilbert space $L^2(\mathbf{R})$, respectively.

5.2 Example of operators on $\mathbf{C}^2$

The most general two–level system in quantum mechanics is described by a complex 2×2 matrix of the form

$$H = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} , \quad (a, b, c \in \mathbf{C}) .$$

For generic values a, b and c, the operator H is not Hermitian, nor normal, nor $\mathcal{PT}$–symmetric. However, in the case where its determinant is real and non-vanishing, the operator H is pseudo–Hermitian and diagonalizable. For such a pseudo–Hermitian Hamiltonian, we now consider the special case

$$H_1 = \begin{bmatrix} 0 & \mathrm{i} \\ -\mathrm{i}\omega^2 & 0 \end{bmatrix} , \tag{19}$$

where the real parameter ω satisfies $\omega > 0$, $\omega \neq 1$ and may be assumed to be time–dependent. The operator H_1 represents the matrix associated to the system of two first order differential equations which is equivalent to the second order differential equation $\ddot{x}(t) + \omega^2 x(t) = 0$ (i.e. the classical equation of motion of the harmonic oscillator).

Let us now display the relevant quantities associated to the operator H_1. This operator is not Hermitian, but it is *quasi–Hermitian* since it admits the real eigenvalues $E_1 = -\omega$ and $E_2 = \omega$. A biorthogonal basis of eigenvectors and associated metric (in the canonical form (10)) are given by[6])

$$|\psi_1\rangle = \begin{bmatrix} -\mathrm{i} \\ \omega \end{bmatrix} , \quad |\psi_2\rangle = \begin{bmatrix} \omega \\ -\mathrm{i}\omega^2 \end{bmatrix} , \quad |\phi_1\rangle = \frac{1}{2} \begin{bmatrix} -\mathrm{i} \\ \omega^{-1} \end{bmatrix} , \quad |\phi_2\rangle = \frac{1}{2} \begin{bmatrix} \omega^{-1} \\ -\mathrm{i}\omega^{-2} \end{bmatrix} ,$$

$$\eta \equiv |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| = \frac{1}{4\omega^2} \begin{bmatrix} 1+\omega^2 & -\mathrm{i}\omega(1-\omega^{-2}) \\ \mathrm{i}\omega(1-\omega^{-2}) & 1+\omega^{-2} \end{bmatrix} . \tag{20}$$

Since the metric operator η is strictly positive, it can be decomposed as

$$\eta = \mathcal{O}^\dagger \mathcal{O} , \quad \text{with} \;\; \mathcal{O} = \frac{1}{2\omega} \begin{bmatrix} \mathrm{i} & -\omega^{-1} \\ \omega & -\mathrm{i} \end{bmatrix} .$$

Thereby, the Hermitian operator associated to the quasi–Hermitian operator H_1 by virtue of equation (18), i.e. $\widehat{H_1} \equiv \mathcal{O} H_1 \mathcal{O}^{-1}$ reads as

$$\widehat{H_1} = H_2 , \quad \text{with} \;\; H_2 \equiv \begin{bmatrix} \omega & 0 \\ 0 & -\omega \end{bmatrix} = \omega\sigma_3 . \tag{21}$$

The latter Hamiltonian describes the interaction of the magnetic moment of a spin–$\frac{1}{2}$ particle with a constant magnetic field.

The Hamiltonian H_1 admits the antilinear symmetry (17) which can presently be rewritten as

$$\chi = \sum_{n=1}^{2} |\psi_n\rangle\langle\tilde{\phi}_n| \, \mathcal{T} , \quad \text{with} \;\; \langle\tilde{\phi}| \equiv (|\phi\rangle)^t \quad (\text{transpose of } |\phi\rangle \in \mathbf{C}^2) . \tag{22}$$

It is explicitly given by

$$\chi = -\mathrm{i} \begin{bmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{bmatrix} \mathcal{T} . \tag{23}$$

[6]) The explicit expressions for the metrics given within reference [6] in equation (67) and in the lines that follow appear to be incorrect and inconsistent with the definitions given earlier in [6] in eqs. (56), (57).

Since $[H, \chi] = 0$, the operator $\widehat{\chi} \equiv \mathcal{O}\chi\mathcal{O}^{-1} = -\mathcal{T}$ represents an antilinear symmetry of the Hermitian operator $\widehat{H}_1 = H_2$. We note that for $\omega = 1$, the operator H_1 is Hermitian and the symmetry (23) then becomes $\chi = -\mathrm{i}\mathcal{PT}$.

In summary, in quantum mechanics, the coupling of the spin–$\frac{1}{2}$ particle to a constant magnetic field can either be described by the self-adjoint and $\mathcal{T}$–invariant operator H_2 (and the standard scalar product of $\mathbf{C}^2$) or by the non-Hermitian operator H_1 which is quasi–Hermitian with respect to the metric (20) (this metric depending on the parameter ω which defines the Hamiltonian).

5.3 Example of operators on $L^2(\mathbf{R})$ with a discrete spectrum

Consider the particular class of non-Hermitian Hamiltonians

$$H = -\partial_x^2 + \beta x^2 + \mathrm{i}(\alpha x^3 + \gamma x)\,, \quad \text{with} \;\; \alpha, \beta, \gamma \in \mathbf{R}\,,\; \alpha \neq 0\,,\; \alpha\gamma \geq 0\,, \tag{24}$$

as acting on smooth functions which tend to zero exponentially as $|x| \to \infty$. The spectrum of H is purely discrete and consists of an infinite number of simple eigenvalues. Quite remarkably, all of these eigenvalues are *real* positive numbers (as proven in reference [22] following an earlier conjecture of Bessis and Zinn–Justin which has been supported by numerical calculations [2]).

The Hamiltonian (24) is $\mathcal{PT}$–symmetric. It is also $\mathcal{PT}$–symmetric if $\alpha > 0$ and $\gamma < 0$, but in this case non-real eigenvalues appear [22]. This illustrates the fact that the mere presence of $\mathcal{PT}$–invariance does not ensure the reality of the spectrum for non-Hermitian Hamiltonians [3].

A special case of the family of Hamiltonians (24) is the harmonic oscillator with an imaginary cubic potential:

$$H_\alpha = -\partial_x^2 + \omega^2 x^2 + \mathrm{i}\alpha x^3\,, \quad (\omega > 0\,,\; \alpha \in \mathbf{R} - \{0\})\,. \tag{25}$$

This Hamiltonian (acting on smooth functions with compact support) is an operator with compact resolvent [19]. This accounts for the discreteness of the spectrum (see appendix), but not for its reality. Since the eigenfunctions of H_α are not known, one cannot apply the given theorem concerning quasi–Hermitian operators. One readily verifies that H_α is pseudo–Hermitian with respect to the metric operator $\eta = \mathcal{P}$ (which is not strictly positive).

5.4 Example of operators on $L^2(\mathbf{R})$ with a continuous spectrum

Consider the Hamiltonian

$$H = -\partial_x^2 + V\,, \quad \text{with} \;\; V(x) = a\mathrm{e}^{\mathrm{i}x} \quad (a \in \mathbf{C}) \tag{26}$$

and a dense domain of definition for H (given by an appropriate Sobolev subspace of $L^2(\mathbf{R})$). Thus, we have a potential which is *complex–valued*, bounded, continuous and *periodic* with period 2π. More generally, one can consider

$$V(x) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{\mathrm{i}nx} \quad \text{with complex coefficients } a_1, a_2, \dots \text{ satisfying} \quad \sum_{n=1}^{\infty} |a_n| < \infty\,. \tag{27}$$

A remarkable result proven by Gasymov in 1980 [23] is that the spectrum of the Hamiltonian (26) involving the potential $V(x) = a\mathrm{e}^{\mathrm{i}x}$ (or more generally the potential (27)) is given by the real half–line $[0, \infty[$ (see also reference [24] for an alternative and elementary proof). Hence, for $V \neq 0$ we have a non-Hermitian Hamiltonian with a real, non-negative, purely continuous spectrum. It is worth noting that this spectrum coincides with the spectrum of the self-adjoint Hamiltonian which describes a free particle ($V = 0$) and that there exists a whole family of complex–valued potentials yielding the same spectrum. If one restricts the study of the given 2π–periodic potentials to the basic interval $[0, 2\pi]$ and imposes Dirichlet or Neumann boundary conditions, the spectrum of the Hamiltonian becomes purely discrete [25].

5.5 Factorization and isospectrality of pseudo−Hermitian operators

As is well known, an operator H which is *self-adjoint and positive* can be factorized according to $H = L^\dagger L$, where L is a linear operator. By replacing $L^\dagger$ by $L^\sharp$ (where the operation $\sharp$ is defined using a metric operator that is not necessarily strictly positive), the given factorization can be generalized to operators H which are only *pseudo–Hermitian* and not necessarily positive. In particular, it then applies to *generic* self-adjoint operators. For more details concerning the following results, see reference [6].

Theorem 5.3 (Factorization of pseudo−Hermitian operators) *Suppose the operator H is pseudo–Hermitian and diagonalizable. Then, there exists a linear operator L on $\mathcal{H}$ (whose pseudo–adjoint $L^\sharp \equiv \eta_1^{-1} L^\dagger \eta_2$ is defined in terms of a pair of metrics η_1, η_2 on $\mathcal{H}$ with respect to which H is pseudo–Hermitian) such that*

$$H = L^\sharp L\,. \tag{28}$$

Example:

The Hamiltonian H_1 introduced in equation (19) is quasi–Hermitian, because it is pseudo–Hermitian with respect to the strictly positive metric (20). It is also pseudo–Hermitian with respect to the following metric (see the comments made in the last footnote)

$$\eta' = -|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| = \frac{1}{4\omega^2}\begin{bmatrix} 1-\omega^2 & \mathrm{i}\omega^{-1}(1+\omega^2) \\ -\mathrm{i}\omega^{-1}(1+\omega^2) & -1+\omega^{-2} \end{bmatrix},$$

which is not positive (e.g. $\langle\psi_1, \eta'\psi_1\rangle = -1$) and whose inverse is given by $(\eta')^{-1} = -|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|$. The operator H_1 (which is neither Hermitian nor positive) decomposes as

$$H_1 = L_1^\sharp L_1 \quad \text{with} \quad \begin{cases} L_1 = \sqrt{\omega}\,\big(|\psi_1\rangle\langle\phi_1| + |\psi_2\rangle\langle\phi_2|\big) = \sqrt{\omega}\,\mathbb{1} \\ L_1^\sharp \equiv (\eta')^{-1} L_1^\dagger \eta = \sqrt{\omega}\,\big(-|\psi_1\rangle\langle\phi_1| + |\psi_2\rangle\langle\phi_2|\big) = \dfrac{1}{\sqrt{\omega}}\, H_1\,. \end{cases}$$

Since $L_1 \propto \mathbb{1}$, the operator H_1 can also be written as $H_1 = L_1 L_1^\sharp$.

The previous theorem is a corollary of the following result.

Theorem 5.4 (Characterization of isospectrality for pseudo−Hermitian operators) *Consider two operators H_1 and H_2 acting on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Suppose H_1 and H_2 are pseudo–Hermitian and diagonalizable. Then the operators H_1 and H_2 are isospectral (except possibly for the eigenvalue zero) and have the same degeneracy structure for the non-vanishing eigenvalues* if and only if *there exists a linear operator $L : \mathcal{H}_1 \to \mathcal{H}_2$ and metrics $\eta_i : \mathcal{H}_i \to \mathcal{H}_i$ (for $i \in \{1, 2\}$) such that*

$$H_1 = L^\sharp L \quad \text{and} \quad H_2 = LL^\sharp \,, \tag{29}$$

where $L^\sharp \equiv \eta_1^{-1} L^\dagger \eta_2$.

Note that the operators H_1 and H_2 as given by the expressions (29) are pseudo–Hermitian with respect to the metrics η_1 and η_2, respectively. Moreover, they are related by the **intertwining relation**

$$LH_1 = H_2 L \tag{30}$$

and its pseudo–adjoint, $L^\sharp H_2 = H_1 L^\sharp$.

Example:

Consider the pair of Hamiltonians H_1, H_2 introduced in equations (19) and (21), respectively. Since the operator H_2 is Hermitian, it is quasi–Hermitian with respect to the metric $\eta_2 = 1\!\!1$. Moreover, as a diagonal matrix, it admits the canonical basis of $\mathbf{C}^2$ as biorthogonal basis of eigenvectors associated to its eigenvalues $E_1 = -\omega$ and $E_2 = \omega$:

$$|\psi_1^{(2)}\rangle = |\phi_1^{(2)}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad |\psi_2^{(2)}\rangle = |\phi_2^{(2)}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The operator H_1 is quasi–Hermitian with respect to the metric $\eta_1 = \eta$ introduced in equation (20). We now have (29) with

$$L = \sqrt{\omega} \left(|\psi_1^{(2)}\rangle\langle\phi_1| + |\psi_2^{(2)}\rangle\langle\phi_2| \right) = \frac{\sqrt{\omega}}{2} \begin{bmatrix} \omega^{-1} & \mathrm{i}\omega^{-2} \\ \mathrm{i} & \omega^{-1} \end{bmatrix}, \tag{31}$$

$$L^\sharp \equiv \eta_1^{-1} L^\dagger \eta_2 = \sqrt{\omega} \left(-|\psi_1\rangle\langle\psi_1^{(2)}| + |\psi_2\rangle\langle\psi_2^{(2)}| \right) = \sqrt{\omega} \begin{bmatrix} \omega & \mathrm{i} \\ -\mathrm{i}\omega^2 & -\omega \end{bmatrix}.$$

6 Pseudo−supersymmetry

From the previous discussion, it follows that standard supersymmetric quantum mechanics (SUSYQM) can be generalized by replacing $\dagger$ by $\sharp$ and, in particular, by considering pseudo–Hermitian operators rather than self-adjoint ones [3, 6]. Thus, the formulation and study of SUSYQM as presented for instance in reference [26] can immediately be transcribed. Here, we only spell out the basic definition and a simple example [6].

Definition: *Let $(\mathcal{H}, H)$ be a pseudo–Hermitian quantum mechanical system with a given metric η. This system is called* ***pseudo−supersymmetric*** *if there exists a bounded self-adjoint operator K (called* involution*) and a finite number of η–pseudo–Hermitian operators $Q_1, \ldots, Q_n$ (called* ***real pseudo−supercharges****), all of which operators act on $\mathcal{H}$ and satisfy $K^2 = \mathbb{1}$, $[K, \eta] = 0$ and*

$$\begin{aligned} \{K, Q_i\} &= 0 && \text{for} \quad i \in \{1, \ldots, n\}\,, \\ \{Q_i, Q_j\} &= 2\delta_{ij} H && \text{for} \quad i, j \in \{1, \ldots, n\}\,. \end{aligned} \tag{32}$$

As in standard SUSYQM [26], the involution K induces a direct sum decomposition of the Hilbert space $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_b \oplus \mathcal{H}_f \quad \text{with} \quad \begin{cases} \mathcal{H}_b = \{\varphi \in \mathcal{H} \mid K\varphi = +\varphi\}\,, \\ \mathcal{H}_f = \{\varphi \in \mathcal{H} \mid K\varphi = -\varphi\}\,. \end{cases} \tag{33}$$

The vectors belonging to $\mathcal{H}_b$ and $\mathcal{H}_f$ are called, respectively, *bosonic* (or *even*) and *fermionic* (or *odd*) vectors.

It is convenient to introduce a matrix notation for the vectors φ belonging to the direct sum (33): $\varphi = \begin{bmatrix} \varphi_b \\ \varphi_f \end{bmatrix}$. With this notation for the vectors, the operator K reads as

$$K = \begin{bmatrix} \mathbb{1}_b & 0 \\ 0 & -\mathbb{1}_f \end{bmatrix}, \tag{34}$$

where $\mathbb{1}_b$ denotes the restriction of the identity operator to the subspace $\mathcal{H}_b$ of $\mathcal{H}$, and analogously for $\mathbb{1}_f$. Since the metric operator η commutes with the involution, it has the diagonal format

$$\eta = \begin{bmatrix} \eta_+ & 0 \\ 0 & \eta_- \end{bmatrix},$$

where $\eta_+ : \mathcal{H}_b \to \mathcal{H}_b$ and $\eta_- : \mathcal{H}_f \to \mathcal{H}_f$ are operators which are everywhere defined, bounded, Hermitian and invertible. The supercharges Q_i have the form

$$Q_i = \begin{bmatrix} 0 & A_i^\sharp \\ A_i & 0 \end{bmatrix}, \tag{35}$$

where A_i is a linear operator and $A_i^\sharp \equiv \eta_+^{-1} A_i^\dagger \eta_-$ represents the restriction of the operation $\sharp$ to operators mapping $\mathcal{H}_b$ to $\mathcal{H}_f$. The resulting Hamiltonian is given by

$$H \equiv \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix} = \begin{bmatrix} A_1^\sharp A_1 & 0 \\ 0 & A_1 A_1^\sharp \end{bmatrix} = \ldots = \begin{bmatrix} A_n^\sharp A_n & 0 \\ 0 & A_n A_n^\sharp \end{bmatrix}, \tag{36}$$

where $A_1, \ldots, A_n$ further satisfy

$$\left.\begin{aligned} A_i^\sharp A_j + A_j^\sharp A_i = 0 \\ A_i A_j^\sharp + A_j A_i^\sharp = 0 \end{aligned}\right\} \quad \text{for} \quad i \neq j\,. \tag{37}$$

The **pseudo−superpartner Hamiltonians** H_+ and H_- are pseudo–Hermitian with respect to the metrics η_+ and η_-, respectively.

According to theorem 3.3, the partner operators H_+ and H_- are isospectral (except possibly for the eigenvalue zero). However, by contrast to ordinary supersymmetry (i.e. self-adjoint Hamiltonians), the energy spectrum is not necessarily positive in the case of pseudo–supersymmetry. Indeed,

$$\langle \varphi, H\varphi \rangle = \langle \varphi, Q_i^2 \varphi \rangle = \langle Q_i^\dagger \varphi, Q_i \varphi \rangle$$

and since Q_i is a priori not self-adjoint, one cannot conclude that the operator H is positive.

Example:

If we consider $A_1 = L$ where L is the operator defined in equation (31), the pseudo–superpartner Hamiltonians are given by expressions (19) and (21):

$$H_+ \equiv L^\sharp L = H_1 \, , \quad H_- \equiv L L^\sharp = H_2 \, .$$

They are completely isospectral, but not positive.

If n is even, say $n = 2m$, we can combine the real supercharges $Q_1, \dots, Q_n$ into **complex supercharges** $\mathbf{q}_1, \dots, \mathbf{q}_m$ given by operators which are not pseudo–Hermitian:

$$\mathbf{q}_i = \frac{1}{\sqrt{2}} \left(Q_{2i-1} + \mathrm{i} Q_{2i} \right) \quad \text{for} \quad i \in \{1, \dots, m\} \, .$$

The cases $n = 2$ and $n = 1$ are equivalent by virtue of the relation $Q_2 = \pm \mathrm{i} K Q_1$ (see reference [26]), and the associated complex supercharge $\mathbf{q} \equiv \frac{1}{\sqrt{2}} (Q_1 + \mathrm{i} Q_2)$ reads as

$$\mathbf{q} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & A^\sharp \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \mathbf{q} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} ,$$

where A is a generic linear operator. The latter can eventually be decomposed into its real and imaginary pseudo–Hermitian parts:

$$A = a_1 + \mathrm{i} a_2 \, , \quad \text{with} \quad \begin{cases} a_1 = \frac{1}{2} (A + A^\sharp) \, , & a_1^\sharp = a_1 \, , \\ a_2 = -\frac{\mathrm{i}}{2} (A - A^\sharp) \, , & a_2^\sharp = a_2 \, . \end{cases} \tag{38}$$

7 Concluding comments

The formalism of *diagonalizable pseudo–Hermitian* operators is certainly well adapted to the investigation of the spectrum of non-Hermitian Hamiltonians acting on a Hilbert space of *finite* dimension. Although the basic definitions and general results can be formulated for Hilbert spaces of infinite dimension, their practical value for concrete examples seems to be limited as long as the basic properties cannot be re-expressed in terms of simple criteria which can be checked in a straightforward way (very much like von Neumann's theory of deficiency indices for studying the self-adjointness of Hermitian operators). We note that some criteria for the existence or for the non-existence of complex eigenvalues of non-Hermitian Hamiltonians which admit a particular linear symmetry have quite recently been put forward [27].

Whatever the *physical* reality of non-Hermitian Hamiltonians and whatever their relevance for quantum mechanics or quantum field theory, there is no doubt that the raised questions are natural, puzzling and challenging.

Appendix: Operators on an infinite dimensional Hilbert space

In the following, we gather some general results concerning operators with a discrete spectrum and concerning spectral decompositions of operators. These facts should prove useful for investigating to which extent the results of subsection 4.1 can be generalized to an infinite dimensional Hilbert space.

For a closed operator H, the spectra of H and $H^\dagger$ related by complex conjugation [14]. In particular, the complex conjugate of an isolated eigenvalue E_n with *finite* multiplicity of the operator H is an eigenvalue of $H^\dagger$ with the same algebraic and geometric multiplicities.

The largest class of bounded operators which behave like matrices representing operators on a finite dimensional Hilbert space are the so-called **compact** or **completely continuous** operators [28]. By definition, an operator H on the Hilbert space $\mathcal{H}$ is compact if the image $\{H\varphi_n\}_{n\in\mathbf{N}}$ of any bounded sequence $\{\varphi_n\}_{n\in\mathbf{N}}$ in $\mathcal{H}$ contains a convergent subsequence. These operators are necessarily bounded and they admit the *canonical decomposition* (12), e.g. see reference [11]. In fact, in the case of an infinite dimensional Hilbert space, the spectrum of a compact operator consists of the point 0 and at most countable many nonzero eigenvalues of finite multiplicity (with the only possible accumulation point at zero).

For a bounded or unbounded **self-adjoint** operator, one has a spectral decomposition of the form (13) involving real spectral values and involving an integral if the spectrum of the operator contains a continuous part.

Just like the self-adjoint operators, the **normal** ones only have point and continuous spectra, but no residual spectrum. They admit a spectral decomposition which is quite analogous to the one of self-adjoint operators, the only difference being that the spectral values and the spectral measure are complex [29, 30]. For normal operators which are compact, one recovers relation (13) involving an infinite sum. A normal operator admitting a real spectrum is necessarily self-adjoint [30].

A class of unbounded operators which have spectra that are analogous to the spectra of operators on a finite dimensional space is provided by the operators with compact resolvent [14, 31]. By definition, a closed Hilbert space operator H is said to be an **operator with compact resolvent** if the associated resolvent operator $R_H(E) = (H - E\mathbb{1})^{-1}$ exists and is compact for some value $E \in \mathbf{C}$ (and thereby for all $E \in \mathbf{C}$). The spectrum of such an operator consists entirely of eigenvalues E_n of finite multiplicity, accumulating only at ∞ in the E–plane (i.e. a *purely discrete spectrum*). On an infinite dimensional Hilbert space, an operator with compact resolvent is necessarily unbounded. (Actually, many differential operators appearing in physics, in particular in connection with classical boundary problems, are of this type.) For a *non-self-adjoint operator with compact resolvent*, a complete set of eigenvectors can be obtained if the following two conditions are satisfied [31]. First, the operator H has to have certain further properties, e.g. its eigenvalues E_n become more and more widely separated as $n \to \infty$, or its eigenvalues lie near the real axis so that H resembles a self-adjoint operator, whose eigenvalues are real. Second, one has to include higher order eigenvectors corresponding to the Jordan blocks for non-diagonalizable matrices.

Finally, we note that a typical non-Hermitian Schrödinger operator (acting on $L^2(\mathbf{R}^n)$) is given by $H = \frac{1}{2m}\,\vec{p}^{\,2} + V(\vec{x})$ where the complex–valued potential $V \equiv V_0 + \mathrm{i}V_1$ is locally integrable on $\mathbf{R}^n$. If

$$V_0(\vec{x}) \geq c > 0 \quad \text{and} \quad |V_1(\vec{x})| \leq q + bV_0(\vec{x}) \quad (q \geq 0\,,\ b \geq 0)\,,$$

a natural domain of definition can be given for the operator H. If, in addition, $V_0(\vec{x})$ tends infinity as $|\vec{x}|$ goes to infinity, the spectrum of H is discrete — see §20 of reference [32] and references given therein for further discussion and in particular for some results concerning the completeness of systems of generalized eigenvectors. We only mention that various completeness criteria exist for **non-self-adjoint operators that are close to self-adjoint ones** [32, 33] and thereby fit into the framework of perturbation theory [19].

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