Quantum (anti)de Sitter algebras and generalizations of the kappa−Minkowski space

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We present two different quantum deformations for the (anti)de Sitter algebras and groups. The former is a non-standard (triangular) deformation of $SO(4,2)$ realized as the conformal group of the (3+1)D Minkowskian spacetime, while the latter is a standard (quasitriangular) deformation of both $SO(2,2)$ and $SO(3,1)$ expressed as the kinematical groups of the (2+1)D anti-de Sitter and de Sitter spacetimes, respectively. The Hopf structure of the quantum algebra and a study of the dual quantum group are presented for each deformation. These results enable us to propose new non-commutative spacetimes that can be interpreted as generalizations of the κ–Minkowski space, either by considering a variable deformation parameter (depending on the boost coordinates) in the conformal deformation, or by introducing an explicit curvature/cosmological constant in the kinematical one; κ–Minkowski turns out to be the common first–order structure for all of these quantum spaces. Some properties provided by these deformations, such as dimensions of the deformation parameter (related with the Planck length), space isotropy, deformed boost transformations, etc., are also commented.

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1 Introduction

Quantum groups have been applied, from the beginning of their development, in the construction of deformed symmetries of spacetimes [1, 2, 3, 4, 5, 6, 7, 8, 9] that generalize classical kinematics beyond Lie algebras. The deformation deepest studied is the well known κ–Poincaré [1, 5, 6] which, more recently, has been applied in the construction of the so called "doubly special relativity" (DSR) theories theories [10, 11, 12, 13, 14] that make use of two fundamental scales. One is the usual observer–independent velocity scale c, while the other is an observer–independent length scale l_p (Planck length) which is assumed to be related with the deformation parameter. In this way, DSR theories have established a relationship between quantum groups and quantum gravity [15, 16].

From the dual quantum group, when the non-commutative spacetime coordinates $\hat{x}^\mu$ conjugated to the κ–Poincaré translations (momenta) are considered, the non-commutative κ–Minkowski spacetime is found [17, 18, 19, 20, 21]:

$$[\hat{x}^0, \hat{x}^i] = -\frac{1}{\kappa}\,\hat{x}^i\,, \quad [\hat{x}^i, \hat{x}^j] = 0\,. \tag{1}$$

More general structures for quantum Minkowskian spacetimes have been proposed to be [22]:

$$[\hat{x}^\mu, \hat{x}^\nu] = \frac{1}{\kappa}(a^\mu \hat{x}^\nu - a^\nu \hat{x}^\mu)\,, \tag{2}$$

where a^μ is a constant four–vector in the Minkowskian space.

However, in spite of the great activity followed in this field, as far as we know, there is no an explicit proposal for a quantum spacetime with a non-zero cosmological constant, or even, some structure generalizing (2). In other words, if Lorentz symmetry has to be modified at the Planck scale and a non-zero curvature/cosmological constant seems to be physically relevant, it is necessary to study the quantum deformations of the (anti)de Sitter groups. These deformations may provide some deformed relativistic symmetries, for which the deformed Poincaré ones should be recovered through a flat limit/contraction or zero cosmological constant.

The aim of this contribution is to present an overview of some recent results concerning quantum (anti)de Sitter algebras and their dual quantum groups from two different deformations; these moreover lead to generalized κ–Minkowski spaces. The structure of the contribution has two different parts, each of them deals with one specific deformation.

In section 2, we present a non-standard quantum deformation of $so(4,2)$ written as the conformal algebra of the (3+1)D Minkowskian spacetime [23, 24]. The Weyl–Poincaré algebra (isometries plus dilations) remains as a Hopf subalgebra after deformation, and this structure is then used to obtain a quantum group which, in turn, provides a non-commutative Minkowskian spacetime. Such a structure involves the quantum boost parameters in the commutation rules; alternatively this can also be interpreted as generalization of (1) with a variable deformation parameter.

In section 3 we study a standard deformation (of Drinfeld–Jimbo type) for $so(2,2)$ and $so(3,1)$ realized as kinematical algebras of the (2+1)D anti-de Sitter (AdS) and de Sitter (dS) spacetimes [25]. We remark that we deal with 2+1 dimensions as these deformations are not completely constructed yet in the proper (3+1)D case. We introduce the quantum algebras and construct the quantum group through a Weyl quantization of a Poisson–Lie structure on the (anti)de Sitter groups. When the non-commutative (anti)de Sitter spacetimes are obtained, it is shown that they can be interpreted as a generalization of κ–Minkowski with a non-zero constant curvature (or cosmological constant).

For both types of deformations the deformation parameter is shown to be related with the Planck length and isotropy of the space is preserved. Some comments on Lorenzt invariance and deformed boost transformations are also given.

2 A non-standard deformation of $SO(4,2)$ in conformal basis

The Lie algebra $so(4,2)$ of the group of conformal transformations of the (3+1)D Minkowskian spacetime $\mathbf{M}^{3+1}$ is spanned by the generators of rotations J_i, time P_0 and space P_i translations, boosts K_i, special conformal transformations C_μ and dilations D. The non-vanishing commutation relations of $so(4,2)$ are given by

$$\begin{array}{lll} [J_i,J_j]=\varepsilon_{ijk}J_k\,, & [J_i,K_j]=\varepsilon_{ijk}K_k\,, & [J_i,P_j]=\varepsilon_{ijk}P_k\,,\\ [J_i,C_j]=\varepsilon_{ijk}C_k\,, & [K_i,K_j]=-\varepsilon_{ijk}J_k\,, & [K_i,P_i]=P_0\,,\\ [K_i,P_0]=P_i\,, & [K_i,C_0]=C_i\,, & [K_i,C_i]=C_0\,,\\ [P_0,C_0]=-2D\,, & [P_0,C_i]=2K_i\,, & [C_0,P_i]=2K_i\,,\\ [P_i,C_j]=2(\delta_{ij}D-\varepsilon_{ijk}J_k)\,, & [D,P_\mu]=P_\mu\,, & [D,C_\mu]=-C_\mu\,, \end{array} \tag{3}$$

where throughout this section we will assume sum over repeated indices, latin indices $i,j,k=1,2,3$, greek indices $\mu,\nu=0,1,2,3$, and a system of units such that $c=\hbar=1$; a generator with three components will be denoted $\mathbf{X}=(X_1,X_2,X_3)$.

As is well known, $so(4,2)$ has two remarkable Lie subalgebras:

• $\{\mathbf{J},\mathbf{K},\mathbf{P},P_0\}$ that generate the Poincaré subalgebra $\mathcal{P}$, that is, the algebra of isometries of the spacetime $\mathbf{M}^{3+1}$.

• $\{\mathbf{J},\mathbf{K},\mathbf{P},P_0,D\}$ that span the Weyl–Poincaré subalgebra $\mathcal{WP}$ which is the similitude algebra of $\mathbf{M}^{3+1}$.

Hence we have the Lie algebra embedding $\mathcal{P}\subset\mathcal{WP}\subset so(4,2)$.

Alternatively, $SO(4,2)$ can also be interpreted as the kinematical group of the (4+1)D AdS spacetime $\mathbf{AdS}^{4+1}$. Explicitly let us denote by L_{AB} $(A<B)$ and T_A $(A,B=0,1\ldots,4)$ the Lorentz and translations generators satisfying

$$\begin{array}{l} [L_{AB},L_{CD}]=\eta_{AC}L_{BD}-\eta_{AD}L_{BC}-\eta_{BC}L_{AD}+\eta_{BD}L_{AC}\,,\\ [L_{AB},T_C]=\eta_{AC}T_B-\eta_{BC}T_A\,, \qquad [T_A,T_B]=-\dfrac{1}{R^2}L_{AB}\,, \end{array} \tag{4}$$

such that $\eta=(\eta_{AB})=\mathrm{diag}\,(-1,1,1,1,1)$ is the Lorentz metric associated to $so(4,1)$, L_{0B} are the four boosts in $\mathbf{AdS}^{4+1}$ and R is the AdS radius related with the cosmological constant by $\Lambda=6/R^2$. Then the change of basis defined by $(i=1,2,3)$:

$$\begin{array}{lll} T_0=-\dfrac{1}{2R}\,(C_0+P_0)\,, & T_1=\dfrac{1}{R}\,D\,, & T_{i+1}=\dfrac{1}{2R}\,(C_i+P_i)\,,\\ L_{01}=\dfrac{1}{2}\,(C_0-P_0)\,, & L_{0,i+1}=K_i\,, & L_{1,i+1}=\dfrac{1}{2}\,(C_i-P_i)\,,\\ L_{23}=J_3\,, & L_{24}=-J_2\,, & L_{34}=J_1\,, \end{array} \tag{5}$$

identify the commutation relations (3) with (4).

2.1 Conformal Lie bialgebra

Now we proceed to introduce a quantum deformation of $so(4,2)$ in the conformal basis (3). The cornerstone of our construction is the non-standard or triangular

Hopf algebra that deforms the Lie algebra with generators J_3, J_+ verifying

$$[J_3, J_+] = J_+ \,, \tag{6}$$

and with classical r–matrix [26, 27] (fulfilling the classical Yang–Baxter equation [28])

$$r = zJ_3 \wedge J_+ = z(J_3 \otimes J_+ - J_+ \otimes J_3)\,, \tag{7}$$

where z is the deformation parameter. The corresponding deformed commutator, coproduct and universal quantum R–matrix are given by:

$$\begin{aligned} [J_3, J_+] &= \frac{\mathrm{e}^{zJ_+} - 1}{z}\,, \\ \Delta(J_+) &= 1 \otimes J_+ + J_+ \otimes 1\,, \\ \Delta(J_3) &= 1 \otimes J_3 + J_3 \otimes \mathrm{e}^{zJ_+}\,, \end{aligned} \tag{8}$$

$$\mathcal{R} = \exp\{-zJ_+ \otimes J_3\} \exp\{zJ_3 \otimes J_+\}\,. \tag{9}$$

This structure is, in fact, a Hopf subalgebra of many known non-standard quantum deformations that cover: $sl(2,\mathbb{R}) \simeq so(2,1)$ [29, 30, 31, 32, 33, 34, 35, 36]; $iso(1,1)$, $gl(2)$ and h_4 [37]; the Schrödinger algebra [38]; as well as $so(2,2)$, $so(3,1)$ and $iso(2,1)$ [39]. The quantum algebra (8) also underlies the approach to physics at the Planck scale early introduced in [40, 41]. We recall that the quantum algebra (8) is also known as the Jordanian deformation which is supported by a twist operator [33] which relates the (classical) cocommutative coproduct with the (deformed) non-cocommutative one while keeping non-deformed commutation rules [27].

The remarkable point is that all of the aforementioned quantum algebras share the same formal classical r–matrix (7), Hopf subalgebra (8), twisting element and universal R–matrix (9). In the following we proceed to construct a quantum $so(4,2)$ algebra starting again from (7) and (8).

Let us consider the non-standard classical r–matrix (7) written in the conformal basis through the identification

$$D \equiv J_3\,, \quad P_0 \equiv J_+\,, \quad \tau \equiv -z\,, \tag{10}$$

where τ is now the deformation parameter related to the usual κ and q deformation parameters through $\tau = 1/\kappa = \ln q$. Next if we assume that

$$r = -\tau D \wedge P_0\,, \tag{11}$$

is the classical r–matrix for the whole $so(4,2)$ algebra (3), the corresponding cocommutator δ of a generic generator Y_i (that defines the associated Lie bialgebra) is obtained as $\delta(Y_i) = [1 \otimes Y_i + Y_i \otimes 1, r]$, namely,

$$\begin{aligned} &\delta(P_0) = 0\,, && \delta(P_i) = \tau P_i \wedge P_0\,, \\ &\delta(J_i) = 0\,, && \delta(D) = -\tau D \wedge P_0\,, \\ &\delta(K_i) = -\tau D \wedge P_i\,, && \delta(C_0) = -\tau C_0 \wedge P_0\,, \\ &\delta(C_i) = -\tau C_i \wedge P_0 + 2\tau D \wedge K_i\,. \end{aligned} \tag{12}$$

The Lie bialgebra $(so(4,2), \delta(r))$ is then formed by the commutation rules (3) and cocommutators (12) and determines the structure of both the quantum algebra and its dual quantum group at the first–order in the deformation parameter, generators and their dual non-commutative group coordinates. In other words, the cocommutator (12) characterizes the first–order quantum group by means of the Lie bialgebra duality [41, 42], that is,

$$\delta(Y_i) = f_i^{jk} Y_j \wedge Y_k \;\Rightarrow\; [\hat{y}^j, \hat{y}^k] = f_i^{jk} \hat{y}^i \,, \tag{13}$$

where $\hat{y}^i$ is the quantum group coordinate dual to Y_i such that $\langle \hat{y}^i | Y_j \rangle = \delta^i_j$. The resulting relations provide the underlying first–order non-commutative spacetime by considering the commutation rules involving the quantum coordinates dual to the translation generators (momenta).

In our case, we denote by $\{\hat{x}^\mu, \hat{\theta}^i, \hat{\xi}^i, \hat{d}, \hat{c}^\mu\}$ the dual non-commutative coordinates of the generators $\{P_\mu, J_i, K_i, D, C_\mu\}$, respectively. Hence from (12) we obtain the following non-vanishing first–order quantum group commutation rules:

$$\begin{aligned} &[\hat{x}^0, \hat{x}^i] = -\tau \hat{x}^i \,, \quad [\hat{x}^0, \hat{d}] = \tau \hat{d} \,, \qquad [\hat{x}^0, \hat{c}^\mu] = \tau \hat{c}^\mu \,, \\ &[\hat{d}, \hat{x}^i] = -\tau \hat{\xi}^i \,, \qquad [\hat{d}, \hat{\xi}^i] = 2\tau \hat{c}^i \,. \end{aligned} \tag{14}$$

Therefore the first–order quantum Minkowskian spacetime, $\mathbf{M}_\tau^{3+1}$, is given by

$$[\hat{x}^0, \hat{x}^i] = -\tau \hat{x}^i \,, \quad [\hat{x}^i, \hat{x}^j] = 0 \,, \tag{15}$$

which coincides exactly with the κ–Minkowski space (1) provided that $\tau = 1/\kappa$. Nevertheless, we shall compute in section 2.3 below the full (all orders) dual quantum group, thus showing in section 2.4 that the complete non-commutative spacetime generalizes the κ–Minkowski space.

On the other hand, the dual map to (5), which relates (14) with a first–order quantum (4+1)D AdS group in the kinematical basis, is given by

$$\begin{aligned} &\hat{t}^0 = -R(\hat{c}^0 + \hat{x}^0) \,, \quad \hat{t}^1 = R\hat{d} \,, \qquad \hat{t}^{i+1} = R(\hat{c}^i + \hat{x}^i) \,, \\ &\hat{l}^{01} = \hat{c}^0 - \hat{x}^0 \,, \qquad \hat{l}^{0,i+1} = \hat{\xi}^i \,, \quad \hat{l}^{1,i+1} = \hat{c}^i - \hat{x}^i \,, \\ &\hat{l}^{23} = \hat{\theta}^3 \,, \qquad\qquad \hat{l}^{24} = -\hat{\theta}^2 \,, \quad \hat{l}^{34} = \hat{\theta}^1 \,, \end{aligned} \tag{16}$$

where $\hat{l}^{AB}$ and $\hat{t}^A$ are, in this order, the non-commutative Lorentz and spacetime coordinates dual to L_{AB} and T_A. Hence the first–order non-vanishing commutation rules for the non-commutative AdS spacetime, $\mathbf{AdS}_\tau^{4+1}$, turn out to be

$$[\hat{t}^0, \hat{t}^1] = -\tau R\, \hat{t}^1 \,, \quad [\hat{t}^0, \hat{t}^{i+1}] = -\tau R^2 \hat{l}^{1,i+1} \,, \quad [\hat{t}^1, \hat{t}^{i+1}] = -\tau R^2 \hat{l}^{0,i+1} \,. \tag{17}$$

Therefore, the maps (5) and (16) (or some kind of non-linear generalization if higher orders in τ and in the coordinates were considered) would allow one to express the same quantum deformation of $so(4,2)$ within two physically different frameworks, thus relating deformed conformal Minkowskian and kinematical AdS symmetries. Such a quantum group relationship might further be applied in order to analyze the role that quantum deformations of $so(4,2)$ could play in relation with the "AdS–CFT correspondence" that relates local QFT on $\mathbf{AdS}^{d+1}$ with a conformal QFT on the compactified Minkowskian spacetime $\mathbf{CM}^{(d-1)+1}$ [43, 44, 45]. We remark that the connection for $d=3$ corresponding to a three–parameter quantum $o(3,2)$ algebra has been studied in [46].

2.2 Quantum conformal algebra

The Hopf structure of the quantum $so(4,2)$ algebra, $U_\tau(so(4,2))$, can be obtained by applying a direct construction [39]. Firstly, we deduce the coproduct Δ as follows.

- Require that (8) remains as a Hopf subalgebra of $U_\tau(so(4,2))$.

- Take into account that the cocommutator δ (12) corresponds to the skewsymmetric part of the first–order deformation of the complete coproduct Δ:

$$\Delta=\sum_{k=0}^{\infty}\Delta_{(k)}=\sum_{k=0}^{\infty}\tau^k\delta_{(k)}\,,\qquad \delta=\delta_{(1)}-\sigma\circ\delta_{(1)}\,, \tag{18}$$

  where $\sigma(X\otimes Y)=Y\otimes X$.

- And solve the coassociativity condition $(1\otimes\Delta)\Delta=(\Delta\otimes 1)\Delta$.

The resulting coproduct turns out to be [23]

$$\begin{aligned}
\Delta(P_0)&=1\otimes P_0+P_0\otimes 1\,,\\
\Delta(P_i)&=1\otimes P_i+P_i\otimes \mathrm{e}^{\tau P_0}\,,\\
\Delta(J_i)&=1\otimes J_i+J_i\otimes 1\,,\\
\Delta(K_i)&=1\otimes K_i+K_i\otimes 1-\tau D\otimes \mathrm{e}^{-\tau P_0}P_i\,,\\
\Delta(D)&=1\otimes D+D\otimes \mathrm{e}^{-\tau P_0}\,,\\
\Delta(C_0)&=1\otimes C_0+C_0\otimes \mathrm{e}^{-\tau P_0}\,,\\
\Delta(C_i)&=1\otimes C_i+C_i\otimes \mathrm{e}^{-\tau P_0}+2\tau D\otimes \mathrm{e}^{-\tau P_0}K_i-\\
&\qquad -\tau^2(D^2+D)\otimes \mathrm{e}^{-2\tau P_0}P_i\,.
\end{aligned} \tag{19}$$

Secondly, the deformed commutation rules are deduced by imposing Δ to be an algebra homomorphism, that is, $\Delta([X,Y])=[\Delta(X),\Delta(Y)]$; these are written in two sets [23]:
• Commutation relations which close a Weyl–Poincaré Hopf subalgebra $U_\tau(\mathcal{WP})\subset$

$U_\tau(so(4,2))$:

$$
\begin{array}{lll}
[J_i,J_j]=\varepsilon_{ijk}J_k\,, & [J_i,K_j]=\varepsilon_{ijk}K_k\,, & [J_i,P_0]=0\,,\\
[J_i,P_j]=\varepsilon_{ijk}P_k\,, & [K_i,K_j]=-\varepsilon_{ijk}J_k\,, & [P_\mu,P_\nu]=0\,,\\
[K_i,P_0]=\mathrm{e}^{-\tau P_0}P_i\,, & [K_i,P_j]=\delta_{ij}\,\dfrac{\mathrm{e}^{\tau P_0}-1}{\tau}\,, & [D,K_i]=0\,,\\
[D,P_i]=P_i\,, & [D,P_0]=\dfrac{1-\mathrm{e}^{-\tau P_0}}{\tau}\,, & [D,J_i]=0\,.
\end{array}
\tag{20}
$$

• Commutation relations that involve the special conformal transformations C_μ:

$$
\begin{array}{ll}
[J_i,C_j]=\varepsilon_{ijk}C_k\,, & [J_i,C_0]=0\,,\\
[C_i,C_j]=0\,, & [C_0,C_i]=-\tau(DC_i+C_iD)\,,\\
[K_i,C_0]=C_i\,, & [K_i,C_j]=\delta_{ij}(C_0-\tau D^2)\,,\\
[P_i,C_j]=2\delta_{ij}D-2\varepsilon_{ijk}J_k\,, & [C_0,P_i]=2K_i+\tau(DP_i+P_iD)\,,\\
[P_0,C_0]=-2D\,, & [P_0,C_i]=\mathrm{e}^{-\tau P_0}K_i+K_i\mathrm{e}^{-\tau P_0}\,,\\
[D,C_i]=-C_i\,, & [D,C_0]=-C_0+\tau D^2\,.
\end{array}
\tag{21}
$$

Finally, the counit and antipode maps can directly be derived from the Hopf algebra axioms and we omit them.

By construction, some relevant Lie subalgebras of $so(4,2)$ are promoted to Hopf subalgebras of $U_\tau(so(4,2))$ after deformation. In particular, we find the following $so(p,q)$ and Weyl–Poincaré Hopf subalgebras, all of them containing the generators P_0 and D, and sharing the same classical r–matrix (11):

$$
\begin{array}{lll}
 & U_\tau(sl(2,\mathbb{R}))\simeq U_\tau(so(2,1)) & \{D,P_0,C_0\}\\
 & \qquad\qquad\qquad\cap & \quad\cap\\
U_\tau(\mathcal{WP}^{1+1})\subset & U_\tau(so(2,2)) & \{D,P_0,P_1,K_1;\ C_0,C_1\}\\
\quad\cap & \quad\cap & \quad\cap\\
U_\tau(\mathcal{WP}^{2+1})\subset & U_\tau(so(3,2)) & \{D,P_0,P_1,P_2,K_1,K_2,J_3;\ C_0,C_1,C_2\}\\
\quad\cap & \quad\cap & \quad\cap\\
U_\tau(\mathcal{WP}^{3+1})\subset & U_\tau(so(4,2)) & \{D,P_0,\mathbf{K},\mathbf{J};\ C_0,\mathbf{C}\}
\end{array}
\tag{22}
$$

However, the Poincaré subalgebras do not remain as Hopf subalgebras after this deformation; this is a consequence of the presence of the dilation generator in the coproduct of the boosts (19). A similar fact was also pointed out for some (standard) Drinfeld–Jimbo deformations in [47].

The chain of embeddings (22) ensures that properties previously known for a given low dimensional deformation can directly be extended to higher dimensional deformations. In this respect, let us consider the universal R–matrix (9) written in the conformal basis by applying the map (10):

$$
\mathcal{R}=\exp\{\tau P_0\otimes D\}\exp\{-\tau D\otimes P_0\}\,. \tag{23}
$$

This element has been shown to be a universal R–matrix for $U_\tau(sl(2,\mathbb{R}))$ [34], that is, this fulfils the quantum Yang–Baxter equation and the property

$$\mathcal{R}\Delta(X)\mathcal{R}^{-1} = \sigma \circ \Delta(X) \quad \text{for} \quad X \in \{D, P_0, C_0\}\,. \tag{24}$$

For the remaining generators of $U_\tau(so(4,2))$, $X \in \{\mathbf{J}, \mathbf{P}, \mathbf{K}, \mathbf{C}\}$, it can be proven that

$$\begin{aligned} &\exp\{-\tau D \otimes P_0\}\Delta(X)\exp\{\tau D \otimes P_0\} = 1 \otimes X + X \otimes 1 \equiv \Delta_0(X)\,,\\ &\exp\{\tau P_0 \otimes D\}\Delta_0(X)\exp\{-\tau P_0 \otimes D\} = \sigma \circ \Delta(X)\,, \end{aligned} \tag{25}$$

so that the element (23) is also the universal R–matrix for $U_\tau(so(4,2))$ as well as for *all* the quantum algebras arising in the embeddings (22).

Another application conveyed by (22) is the extension of the time discretization of the (1+1)D massless Klein–Gordon (or wave) equation [39] associated to $U_\tau(so(2,2))$ to (2+1)D with $U_\tau(so(3,2))$ [48] and to (3+1)D with $U_\tau(so(4,2))$ [23]. The generators of all of these quantum algebras have been realized as differential–difference operators acting on a uniform Minkowskian spacetime lattice discretized along the time direction (the space coordinates remain continuous) and with the deformation parameter τ playing the role of the time lattice constant.

2.3 Quantum Weyl−Poincaré group

The existence of the universal R–matrix (23) enables, in principle, to deduce the quantum group dual to any of the quantum algebras appearing in (22) by applying the Faddeev–Reshetikhin–Takhtajan (FRT) procedure [49]. Such an approach requires a matrix representation of the chosen quantum algebra as well as a matrix element $\hat{\mathcal{T}}$ of the quantum group with non-commutative entries.

In what follows we shall restrict ourselves to deal with the quantum group dual to $U_\tau(\mathcal{WP})$ instead of that dual to the complete $U_\tau(so(4,2))$ since for the latter it is not possible to identify properly the quantum space and time coordinates but only formal non-commutative matrix entries.

The change of basis (5) allows us to deduce a 6×6 deformed matrix representation of $U_\tau(so(4,2))$ in the conformal basis (fulfilling (20) and (21)) by starting from the vector representation of the (4+1)D quantum AdS algebra; namely

$$\begin{aligned} &P_0 = \frac{\tau}{2}\big(e_{00} - e_{01} + e_{10} - e_{11}\big) - e_{02} - e_{12} + e_{20} - e_{21}\,,\\ &P_i = e_{0,i+2} + e_{1,i+2} + e_{i+2,0} - e_{i+2,1}\,, \qquad D = e_{01} + e_{10}\,,\\ &J_i = -\varepsilon_{ijk} e_{j+2,k+2}\,, \qquad K_i = e_{2,i+2} + e_{i+2,2}\,,\\ &C_0 = \tau\big(e_{00} + e_{11}\big) - e_{02} + e_{12} + e_{20} + e_{21}\,,\\ &C_i = e_{0,i+2} - e_{1,i+2} + e_{i+2,0} + e_{i+2,1}\,, \end{aligned} \tag{26}$$

where e_{ab} $(a,b = 0,\dots,5)$ is the 6×6 matrix with entries $(e_{ab})_{ij} = \delta_{ai}\delta_{bj}$. Hence a 6×6 matrix representation for the quantum Weyl–Poincaré algebra (20) arises within (26).

Under this representation P_0^3 vanishes, so that the quantum R–matrix (23) reduces to the 36×36 matrix given by

$$\mathcal{R} = \left(\mathbf{1} \otimes \mathbf{1} + \tau P_0 \otimes D + \tfrac{1}{2}\tau^2 P_0^2 \otimes D^2\right)\left(\mathbf{1} \otimes \mathbf{1} - \tau D \otimes P_0 + \tfrac{1}{2}\tau^2 D^2 \otimes P_0^2\right), \quad (27)$$

where $\mathbf{1}$ is the 6×6 unit matrix.

Next we construct the quantum Weyl–Poincaré group element $\hat{\mathcal{T}}$ by considering the following matrix product that depends on the matrix generators (26) and their dual non-commutative coordinates:

$$\hat{\mathcal{T}} = \mathrm{e}^{\hat{d}D}\,\mathrm{e}^{\hat{x}^0 P_0}\left(\mathrm{e}^{\hat{x}^1 P_1}\,\mathrm{e}^{\hat{x}^2 P_2}\,\mathrm{e}^{\hat{x}^3 P_3}\right)\left(\mathrm{e}^{\hat{\theta}^1 J_1}\,\mathrm{e}^{\hat{\theta}^2 J_2}\,\mathrm{e}^{\hat{\theta}^3 J_3}\right)\left(\mathrm{e}^{\hat{\xi}^1 K_1}\,\mathrm{e}^{\hat{\xi}^2 K_2}\,\mathrm{e}^{\hat{\xi}^3 K_3}\right) =$$
$$= \begin{pmatrix} \hat{\alpha}_+ & \hat{\beta}_- & \hat{\gamma}_0 & \hat{\gamma}_1 & \hat{\gamma}_2 & \hat{\gamma}_3 \\ \hat{\beta}_+ & \hat{\alpha}_- & \hat{\gamma}_0 & \hat{\gamma}_1 & \hat{\gamma}_2 & \hat{\gamma}_3 \\ \hat{x}^0 & -\hat{x}^0 & \hat{\Lambda}^0_0 & \hat{\Lambda}^0_1 & \hat{\Lambda}^0_2 & \hat{\Lambda}^0_3 \\ \hat{x}^1 & -\hat{x}^1 & \hat{\Lambda}^1_0 & \hat{\Lambda}^1_1 & \hat{\Lambda}^1_2 & \hat{\Lambda}^1_3 \\ \hat{x}^2 & -\hat{x}^2 & \hat{\Lambda}^2_0 & \hat{\Lambda}^2_1 & \hat{\Lambda}^2_2 & \hat{\Lambda}^2_3 \\ \hat{x}^3 & -\hat{x}^3 & \hat{\Lambda}^3_0 & \hat{\Lambda}^3_1 & \hat{\Lambda}^3_2 & \hat{\Lambda}^3_3 \end{pmatrix}. \quad (28)$$

The non-commutative entries are just the quantum Minkowskian coordinates $\hat{x}^\mu$, the formal Lorentz entries $\hat{\Lambda}^\mu_\nu = \hat{\Lambda}^\mu_\nu(\hat{\theta}^i, \hat{\xi}^i)$, which involve quantum rotation and boost coordinates, verifying

$$\hat{\Lambda}^\mu_\nu \hat{\Lambda}^\rho_\sigma g^{\nu\sigma} = g^{\mu\rho}\,, \quad \hat{x}_\mu = g_{\mu\nu}\hat{x}^\nu\,, \quad (g^{\mu\rho}) = \mathrm{diag}\,(-1,1,1,1)\,, \quad (29)$$

as well some functions $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ of the quantum coordinates which are defined by

$$\begin{aligned} &\hat{\alpha}_\pm = \cosh\hat{d} \pm \tfrac{1}{2}\,\mathrm{e}^{\hat{d}}\left(\hat{x}_\mu\hat{x}^\mu + \tau\hat{x}^0\right), \quad \hat{\gamma}_\nu = \mathrm{e}^{\hat{d}}\hat{x}_\mu\hat{\Lambda}^\mu_\nu\,, \\ &\hat{\beta}_\pm = \sinh\hat{d} \pm \tfrac{1}{2}\,\mathrm{e}^{\hat{d}}\left(\hat{x}_\mu\hat{x}^\mu + \tau\hat{x}^0\right). \end{aligned} \quad (30)$$

Notice that if the complete quantum $SO(4,2)$ group were considered by adding the remaining exponentials of the conformal transformations (26) to the product (28), the non-commutative coordinates $\hat{x}^\mu$ will no longer appear as themselves, thus precluding a further and direct identification of the associated non-commutative spacetime as usually happens when dealing with quantum deformations of semisimple groups (see, e.g., [50] for the construction of a standard q–$SO(3,2)$).

Now the FRT approach gives rise to the commutation rules, coproduct, counit and antipode by means of the relations

$$\mathcal{R}\hat{\mathcal{T}}_1\hat{\mathcal{T}}_2 = \hat{\mathcal{T}}_2\hat{\mathcal{T}}_1\mathcal{R}\,, \quad \Delta(\hat{\mathcal{T}}) = \hat{\mathcal{T}}\dot{\otimes}\hat{\mathcal{T}}\,, \quad \epsilon(\hat{\mathcal{T}}) = \mathbf{1}\,, \quad S(\hat{\mathcal{T}}) = \hat{\mathcal{T}}^{-1}\,, \quad (31)$$

respectively, and where $\hat{\mathcal{T}}_1 = \hat{\mathcal{T}} \otimes \mathbf{1}$ and $\hat{\mathcal{T}}_2 = \mathbf{1} \otimes \hat{\mathcal{T}}$. The resulting commutation rules and coproduct initially depend on all the entries of $\hat{\mathcal{T}}$, but they can be further and consistently reduced, with the aid of (30), to expressions that only depend on $\{\hat{d}, \hat{x}^\mu, \hat{\Lambda}^\mu_\nu\}$; these are [24]

$$\begin{aligned} &\Delta(\hat{x}^\mu) = \hat{x}^\mu \otimes \mathrm{e}^{-\hat{d}} + \hat{\Lambda}^\mu_\eta \otimes \hat{x}^\eta\,, \\ &\Delta(\hat{d}) = \hat{d} \otimes 1 + 1 \otimes \hat{d}\,, \\ &\Delta(\hat{\Lambda}^\mu_\nu) = \hat{\Lambda}^\mu_\eta \otimes \hat{\Lambda}^\eta_\nu\,, \end{aligned} \quad (32)$$

$$
\begin{aligned}
&[\hat{d},\hat{\Lambda}^\mu_\nu]=0\,, \qquad [\hat{x}^\alpha,\hat{\Lambda}^\mu_\nu]=0\,, \qquad [\hat{\Lambda}^\alpha_\beta,\hat{\Lambda}^\mu_\nu]=0\,,\\
&[\hat{d},\hat{x}^\mu]=\tau\left(\delta^\mu_0\mathrm{e}^{-\hat{d}}-\hat{\Lambda}^\mu_0\right), \qquad [\hat{x}^\mu,\hat{x}^\nu]=\tau\left(\hat{\Lambda}^\nu_0\hat{x}^\mu-\hat{\Lambda}^\mu_0\hat{x}^\nu\right),
\end{aligned}
\tag{33}
$$

where the quantum Lorentz entries $\hat{\Lambda}^\mu_0$ are given by

$$
\begin{aligned}
&\hat{\Lambda}^0_0=\cosh\hat{\xi}^1\cosh\hat{\xi}^2\cosh\hat{\xi}^3\,, \qquad \hat{\Lambda}^2_0=\sinh\hat{\xi}^2\cosh\hat{\xi}^3\,,\\
&\hat{\Lambda}^1_0=\sinh\hat{\xi}^1\cosh\hat{\xi}^2\cosh\hat{\xi}^3\,, \qquad \hat{\Lambda}^3_0=\sinh\hat{\xi}^3\,.
\end{aligned}
\tag{34}
$$

The commutation relations (33) show that the functions $\hat{\Lambda}^\mu_\nu$ are indeed commuting quantities, so that there are no ordering problems in any of the above expressions; this, in turn, implies that $[\hat{\xi}^i,\hat{\xi}^j]=0$.

Notice also that if we take in (33) the first–order in all the quantum coordinates (in this case $\hat{\Lambda}^0_0\to 1$ and $\hat{\Lambda}^i_0\to\hat{\xi}^i$), we recover the relations defining the Weyl–Poincaré bialgebra in its dual form as

$$
[\hat{x}^0,\hat{x}^i]=-\tau\hat{x}^i\,, \quad [\hat{d},\hat{x}^0]=-\tau\hat{d}\,, \quad [\hat{d},\hat{x}^i]=-\tau\hat{\xi}^i\,,
\tag{35}
$$

which coincide with (14) provided that $\hat{c}^\mu\equiv 0$.

2.4 Non-commutative Minkowskian spacetime

The non-commutative Minkowskian spacetime $\mathbf{M}^{3+1}_\tau$ with quantum Weyl–Poincaré group symmetry is identified within the set of commutation rules (33) by considering those involving the quantum coordinates $\hat{x}^\mu$ [24]:

$$
[\hat{x}^\mu,\hat{x}^\nu]=\tau\left(\hat{\Lambda}^\nu_0(\hat{\xi})\hat{x}^\mu-\hat{\Lambda}^\mu_0(\hat{\xi})\hat{x}^\nu\right).
\tag{36}
$$

This can be interpreted as a generalization of (2) through the map $a^\mu\to\hat{\Lambda}^\mu_0(\hat{\xi})$, where the Lorentz entries involved only depend on the quantum boost coordinates as given in (34).

The commutativity character of $\hat{\Lambda}^\mu_0(\hat{\xi})$ shown in (33) suggests that these can be regarded as structure constants in (36). As a byproduct, the quantum boost coordinates $\hat{\xi}^i$ are commutative coordinates (so scalars) within the quantum Weyl–Poincaré group. We would also like to mentioning that if the quantum conformal transformations C_μ and parameters $\hat{c}^\mu$ were taken into account and the corresponding quantum $SO(4,2)$ group were constructed, then $\hat{\xi}^i$ and $\hat{\Lambda}^\mu_0$ no longer would commute with the dilation parameter $\hat{d}$, as follows from the first–order relations (14), in such a manner that (36) would define a quadratic non-commutative Minkowskian spacetime.

An alternative form to express (36), formally closer to κ–Minkowski, is achieved by introducing new quantum space coordinates $\hat{X}^i$ defined by

$$
\hat{x}^0\to\hat{x}^0\,, \quad \hat{x}^i\to\hat{X}^i=\hat{x}^i\hat{\Lambda}^0_0(\hat{\xi})-\hat{x}^0\hat{\Lambda}^i_0(\hat{\xi})\,.
\tag{37}
$$

The transformed $\mathbf{M}_\tau^{3+1}$ reads

$$[\hat{X}^i, \hat{x}^0] = \tau \hat{\Lambda}^0_0(\hat{\xi})\hat{X}^i\,, \quad [\hat{X}^i, \hat{X}^j] = 0\,, \tag{38}$$

which shows a generalization of the κ–Minkowski space (1) with a "variable" deformation parameter $\tau'(\hat{\xi}) = \tau \cosh\hat{\xi}^1 \cosh\hat{\xi}^2 \cosh\hat{\xi}^3$.

Now we shall comment on some of the properties derived from the above results.

2.4.1 Dimensions of the deformation parameter

Dimensional analysis of this deformation (see, e.g., expressions (11) or (35)) shows that the deformation parameter τ is endowed with the same dimensions as the Planck length l_P (recall that we consider units with $c = \hbar = 1$), so inverse to the parameter κ; these are inherited either from P_0 or from $\hat{x}^0$:

$$[\tau] = [P_0]^{-1} = [\hat{x}^0]\,, \quad [\tau] = \frac{1}{[\kappa]}\,. \tag{39}$$

Therefore τ is a dimensionful deformation parameter that can be considered to be related with the Planck length, thus playing the role of an observer–independent (fundamental) scale. In this respect, we remark that $U_\tau(\mathcal{WP})$ has allowed us to construct a DSR theory, different from the proposals coming from κ–Poincaré, which can be found in [51].

2.4.2 Space isotropy

The explicit form of $\mathbf{M}_\tau^{3+1}$ (36) (also (38)) shows that the quantum rotation coordinates $\hat{\theta}^i$ are absent; these only appear as arguments of the quantum Lorentz entries $\hat{\Lambda}^\mu_i = \hat{\Lambda}^\mu_i(\hat{\theta}, \hat{\xi})$. In this sense we can say that the isotropy of the space is preserved.

The same property follows from the Hopf structure of $U_\tau(\mathcal{WP})$. The coproduct (19) exhibits non-deformed (cocommutative) rotation generators (this a direct consequence of the bialgebra (12) since $\delta(J_i) = 0$), while the deformed commutation relations (20) show that $\mathbf{P}$ and $\mathbf{K}$ are transformed as classical vectors under rotations. In fact, $\mathbf{J}$ close a non-deformed $so(3)$ algebra.

2.4.3 Lorentz invariance

The explicit dependence of $\hat{\Lambda}^\mu_0$ on the quantum boost coordinates $\hat{\xi}$ in $\mathbf{M}_\tau^{3+1}$ indicates that different observers in relative motion with respect to quantum group transformations have a different perception of the spacetime non-commutativity which, in turn, implies that Lorentz invariance is lost.

Nevertheless, from our point of view, the required property in the context of quantum groups should be Lorentz coinvariance rather than Lorentz invariance. This means that the commutation rules (36) that define $\mathbf{M}_\tau^{3+1}$ should be coinvariant under quantum group transformations, that is, under the transformation laws for the quantum coordinates which are provided by the coproduct (32). Covariance of

(36) under such quantum group transformations is ensured by construction and we refer to [24] for more details.

2.4.4 Boost transformations

The deformed finite boost transformations have been obtained within the DSR theory developed in [51] by working with the quantum algebra $U_\tau(\mathcal{WP})$. Such transformations close a group as in the non-deformed case and moreover the additivity of the boost parameter (rapidity) for two deformed transformations along a same direction is preserved. In this respect, we remark that these properties are in full agreement with the commutation rule $[\hat{\xi}^i, \hat{\xi}^j] = 0$ provided by the quantum group.

Finally, we also recall that the range of boost parameters, energy P_0 and momenta $\mathbf{P}$ deeply depend on the sign of the deformation parameter τ, so that two different scenarios appear [51].

3 A standard deformation of $SO(2,2)$ and $SO(3,1)$ in kinematical basis

The Lie algebras of the isometry groups of the three (2+1)D relativistic spacetimes of constant curvature ω can be described in a unified way by means of the curvature itself, which plays the role of a graded contraction parameter [7]; we denote this family of Lie algebras by $so_\omega(2,2)$. If $\{J, P_0, P_i, K_i\}$ are, in this order, the generators of rotations, time translations, space translations and boosts, the commutation relations of $so_\omega(2,2)$ read

$$\begin{array}{lll} [J, P_i] = \epsilon_{ij} P_j\,, & [J, K_i] = \epsilon_{ij} K_j\,, & [J, P_0] = 0\,, \\ [P_i, K_j] = -\delta_{ij} P_0\,, & [P_0, K_i] = -P_i\,, & [K_1, K_2] = -J\,, \\ [P_0, P_i] = \omega K_i\,, & [P_1, P_2] = -\omega J\,, & \end{array} \tag{40}$$

where, in contrast with section 2, we now assume that Latin indices $i, j = 1, 2$, Greek ones $\mu, \nu = 0, 1, 2$, and ϵ_{ij} is a skewsymmetric tensor such that $\epsilon_{12} = 1$.

According to the sign of ω we find that the Lie brackets (40) reproduce:

- The AdS algebra, $so(2,2)$, when $\omega = +1/R^2 > 0$ and where R is the AdS radius.
- The dS algebra, $so(3,1)$, when $\omega = -1/R^2 < 0$ and where R is the dS radius.
- And the Poincaré algebra, $iso(2,1)$, when $\omega = 0$; this case also corresponds to the flat limit/contraction $R \to \infty$ such that $so(2,2) \to iso(2,1) \leftarrow so(3,1)$.

3.1 (Anti)de Sitter Lie bialgebras

The quantum deformation we are going to deal with in this section is based in the Drinfeld–Jimbo deformation of $sl(2,\mathbb{R}) \simeq so(2,1)$ [26], $U_z(sl(2,\mathbb{R}))$, whose classical r–matrix, deformed commutation rules and coproduct in the basis $\{J_3, J_\pm\}$ are given by

$$r = z J_+ \wedge J_-\,, \tag{41}$$

$$
\begin{aligned}
&[J_3, J_\pm] = \pm J_\pm \,, \qquad [J_+, J_-] = \frac{\sinh(2zJ_3)}{z} \,, \\
&\Delta(J_3) = 1 \otimes J_3 + J_3 \otimes 1 \,, \\
&\Delta(J_\pm) = \mathrm{e}^{-zJ_3} \otimes J_\pm + J_\pm \otimes \mathrm{e}^{zJ_3} \,.
\end{aligned} \tag{42}
$$

The r–matrix (41) is a solution of the modified classical Yang–Baxter equation, so that $U_z(sl(2,\mathbb{R}))$ is a deformation of standard or quasitriangular type.

The well known Lie algebra isomorphism $so(2,1) \oplus so(2,1) \simeq so(2,2)$ can be implemented in the quantum group framework. As far as the classical r–matrix is concerned, the difference of two copies of (41) with the same deformation parameter gives rise to a classical r–matrix of $so(2,2)$ [52], here written in the kinematical basis

$$
r = z(K_1 \wedge P_1 + K_2 \wedge P_2) \,. \tag{43}
$$

This element is again a solution of the modified classical Yang–Baxter equation, not only for $so(2,2)$ but also for $so(3,1)$ and $iso(2,1)$, no matter of the curvature ω. In fact, this is exactly the r–matrix underlying the (2+1)D κ–Poincaré algebra provided that $z = 1/\kappa$. Hence we can take (43) as the r–matrix for the whole family $so_\omega(2,2)$.

By following the same procedure of section 2.1 we obtain that the cocommutator coming from (43) reads

$$
\begin{aligned}
&\delta(P_0) = 0 \,, \qquad \delta(J) = 0 \,, \\
&\delta(P_i) = z(P_i \wedge P_0 - \omega \epsilon_{ij} K_j \wedge J) \,, \\
&\delta(K_i) = z(K_i \wedge P_0 + \epsilon_{ij} P_j \wedge J) \,.
\end{aligned} \tag{44}
$$

Meanwhile the dual non-vanishing commutation rules turn out to be

$$
\begin{aligned}
&[\hat\theta, \hat x^i] = z\epsilon_{ij}\hat\xi^j \,, \qquad [\hat x^0, \hat x^i] = -z\hat x^i \,, \\
&[\hat\theta, \hat\xi^i] = -z\omega\epsilon_{ij}\hat x^j \,, \quad [\hat x^0, \hat\xi^i] = -z\hat\xi^i \,,
\end{aligned} \tag{45}
$$

where $\{\hat\theta, \hat x^\mu, \hat\xi^i\}$ are the non-commutative group coordinates dual to the generators $\{J, P_\mu, K_i\}$, respectively.

Consequently, from (45) we find that the first–order non-commutative AdS, Minkowskian and dS spacetimes are simultaneously defined by the (2+1)D κ–Minkowski space (similarly to the previous deformation):

$$
[\hat x^0, \hat x^i] = -z\hat x^i \,, \quad [\hat x^1, \hat x^2] = 0 \,. \tag{46}
$$

As it can be expected, when higher orders in the quantum coordinates are considered the resulting non-commutative spaces generalize the κ–Minkowski one since corrections depending on the curvature appear; thus we shall present in section 3.4 below three different quantum spaces, all of them sharing the same first–order relations (46).

3.2 Quantum (anti)de Sitter algebras

Let us consider two copies of the quantum algebra (42) such that the two deformation parameters only differ by the sign. Then the direct sum of quantum algebras [3, 52, 53]

$$U_z(so(2,1)) \oplus U_{-z}(so(2,1)) \simeq U_z(so(2,2))\,, \tag{47}$$

together with a contraction analysis lead to the Hopf structure of the standard quantum deformation of (40) and (44) denoted $U_z(so_\omega(2,2))$ [7]:

$$\begin{aligned}
\Delta(P_0) &= 1\otimes P_0 + P_0\otimes 1\,, \qquad \Delta(J) = 1\otimes J + J\otimes 1\,,\\
\Delta(P_i) &= \mathrm{e}^{-zP_0/2}\cosh(\tfrac{z}{2}\,\rho J)\otimes P_i + P_i\otimes \mathrm{e}^{zP_0/2}\cosh(\tfrac{z}{2}\,\rho J) +\\
&\quad +\rho\mathrm{e}^{-zP_0/2}\sinh(\tfrac{z}{2}\,\rho J)\otimes \epsilon_{ij}K_j - \rho\epsilon_{ij}K_j\otimes \mathrm{e}^{zP_0/2}\sinh(\tfrac{z}{2}\,\rho J)\,,\\
\Delta(K_i) &= \mathrm{e}^{-zP_0/2}\cosh(\tfrac{z}{2}\,\rho J)\otimes K_i + K_i\otimes \mathrm{e}^{zP_0/2}\cosh(\tfrac{z}{2}\,\rho J) -\\
&\quad -\mathrm{e}^{-zP_0/2}\,\frac{\sinh(\tfrac{z}{2}\,\rho J)}{\rho}\otimes \epsilon_{ij}P_j + \epsilon_{ij}P_j\otimes \mathrm{e}^{zP_0/2}\,\frac{\sinh(\tfrac{z}{2}\,\rho J)}{\rho}\,,
\end{aligned} \tag{48}$$

$$\begin{aligned}
&[J,P_i] = \epsilon_{ij}P_j\,, \qquad [J,K_i] = \epsilon_{ij}K_j\,, \qquad [J,P_0] = 0\,,\\
&[P_i,K_j] = -\delta_{ij}\,\frac{\sinh(zP_0)}{z}\,\cosh(z\rho J)\,, \qquad [P_0,K_i] = -P_i\,,\\
&[K_1,K_2] = -\cosh(zP_0)\,\frac{\sinh(z\rho J)}{z\rho}\,, \qquad [P_0,P_i] = \omega K_i\,,\\
&[P_1,P_2] = -\omega\cosh(zP_0)\,\frac{\sinh(z\rho J)}{z\rho}\,,
\end{aligned} \tag{49}$$

where hereafter we also express the curvature as $\omega = \rho^2$. Therefore $\rho = 1/R$ for $U_z(so(2,2))$, $\rho = \mathrm{i}/R$ for $U_z(so(3,1))$, and the contraction $\omega = 0$ to κ–Poincaré $U_z(iso(2,1))$ corresponds to the flat limit $\rho \to 0$. We remark that such a contraction is always well defined in any of the expressions presented in this section, so that it is not necessary to perform any kind of quantum Inönü–Wigner contractions [54] as we deal with a built-in scheme of contractions [7]. In particular, the limit $\rho \to 0$ of $U_z(so_\omega(2,2))$ directly gives rise to the coproduct and deformed commutation rules of κ–Poincaré as

$$\begin{aligned}
\Delta(P_0) &= 1\otimes P_0 + P_0\otimes 1\,, \qquad \Delta(J) = 1\otimes J + J\otimes 1\,,\\
\Delta(P_i) &= \mathrm{e}^{-zP_0/2}\otimes P_i + P_i\otimes \mathrm{e}^{zP_0/2}\,,\\
\Delta(K_i) &= \mathrm{e}^{-zP_0/2}\otimes K_i + K_i\otimes \mathrm{e}^{zP_0/2} -\\
&\quad -\frac{z}{2}\,\epsilon_{ij}\left(\mathrm{e}^{-zP_0/2}J\otimes P_j - P_j\otimes \mathrm{e}^{zP_0/2}J\right),
\end{aligned} \tag{50}$$

$$\begin{aligned}
&[J,P_i] = \epsilon_{ij}P_j\,, \qquad [J,K_i] = \epsilon_{ij}K_j\,, \qquad [J,P_0] = 0\,,\\
&[P_i,K_j] = -\delta_{ij}\,\frac{\sinh(zP_0)}{z}\,, \qquad [P_0,K_i] = -P_i\,,\\
&[K_1,K_2] = -J\cosh(zP_0)\,, \qquad [P_\mu,P_\nu] = 0\,.
\end{aligned} \tag{51}$$

3.3 Quantum (anti)de Sitter groups: a Poisson−Lie structure

As we have already commented in the section 2.3 for $U_\tau(so(4,2))$, to obtain the complete quantum group dual to a quantum deformation of a semisimple Lie algebra is, in general, a cumbersome task that requires to know a matrix representation of both the quantum group element $\hat{\mathcal{T}}$ and the quantum R–matrix $\mathcal{R}$; the FRT procedure can then be applied. However, even if these objects are known, the quantum spacetime and boost coordinates would appear "hidden" as arguments of some formal non-commutative entries of the matrix $\hat{\mathcal{T}}$.

Another way to get an insight into the non-commutative structures associated to the quantum group is to compute the Poisson–Lie brackets provided by the classical r–matrix for the commutative coordinates, and next to analyzing their possible non-commutative version. The steps of this procedure are as follows.

- Obtain a matrix element $\mathcal{T}$ of the Lie group (so with commutative coordinates) by means of a matrix representation of the algebra.
- From $\mathcal{T}$, compute left Y_i^L and right Y_i^R invariant vector fields for each algebra generator Y_i.
- Construct the Poisson–Lie structure on the group that comes from the classical r–matrix $r = r^{ij} Y_i \otimes Y_j$ through the Sklyanin bracket defined by [28]:

$$\{f,g\} = r^{ij}(Y_i^L f\, Y_j^L g - Y_i^R f\, Y_j^R g)\,, \tag{52}$$

  where f, g are smooth functions of the Lie group coordinates y^i. In this way the Poisson–Lie brackets for y^i can be obtained, say

$$\{y^i, y^j\} = zF(y^k)\,, \tag{53}$$

  where $F(y^k)$ is a smooth function depending on some set of coordinates y^k.
- Finally, apply the Weyl substitution of the initial Poisson brackets between commutative coordinates (53) by commutators between non-commutative coordinates [28, 55].

$$[\hat{y}^i, \hat{y}^j] = zF(\hat{y}^k) + o(z^2)\,. \tag{54}$$

Obviously, there is no guarantee that the Weyl quantization gives the complete quantum group dual to the initial quantum algebra, specially when dealing with semisimple groups, since ordering problems often appear during the quantization procedure. However, this approach provides, at least, the non-commutative structure up to second–order in the deformation parameter and in *all* orders in the quantum coordinates; note that the $o(z^2)$ term in (54) comes from the reordering of the quantum coordinates $\hat{y}^k$ within the function F. Recall that the κ–Poincaré group [17, 18, 19, 20] (in any dimension) has been constructed by applying the above procedure. In fact, for this deformation there does not exist a universal R–matrix except for the (2+1)D case [17, 53, 56].

In our case, we start with the following 4D real matrix representation of $so_\omega(2,2)$ (verifying (40)):

$$\begin{aligned} &P_0 = -\omega e_{01} + e_{10}\,, \qquad P_i = \omega e_{0\,i+1} + e_{i+1,0}\,,\\ &J = -e_{23} + e_{32}\,, \qquad\quad K_i = e_{1\,i+1} + e_{i+1,1}\,, \end{aligned} \tag{55}$$

where e_{ab} $(a,b=0,\dots,3)$ is the 4×4 matrix with entries $(e_{ab})_{ij} = \delta_{ai}\delta_{bj}$. Under this representation, any generator $Y \in so_\omega(2,2)$ fulfils

$$Y^T\mathbb{I} + \mathbb{I}Y = 0\,, \quad \mathbb{I} = \mathrm{diag}\,(1,\omega,-\omega,-\omega)\,, \tag{56}$$

where Y^T is the transpose matrix of Y. Next we construct a 4×4 matrix element of the group $SO_\omega(2,2)$, under the representation (55), through the following product:

$$T = \exp(x^0P_0)\exp(x^1P_1)\exp(x^2P_2)\exp(\xi^1K_1)\exp(\xi^2K_2)\exp(\theta J)\,. \tag{57}$$

Left and right invariant vector fields, Y^L and Y^R, of $SO_\omega(2,2)$ and the Poisson–Lie structure associated to the r–matrix (43) can then be computed and they can explicitly be found in [25]. We only present here the Poisson–Lie brackets involving group spacetime x^μ and boost ξ^i coordinates:

$$\{x^0,x^1\} = -z\,\frac{\tanh\rho x^1}{\rho\cosh^2\rho x^2}\,, \quad \{x^0,x^2\} = -z\,\frac{\tanh\rho x^2}{\rho}\,, \quad \{x^1,x^2\} = 0\,, \tag{58}$$

$$\{\xi^1,\xi^2\} = z\rho\,\frac{\sinh\rho x^1}{\cosh\rho x^2}\left(\frac{\sinh\rho x^1\tanh\rho x^2\sinh\xi^1 + \cosh\xi^1\sinh\xi^2}{\cosh\rho x^1} - \frac{\tanh\xi^2}{\cosh\rho x^2}\right), \tag{59}$$

and the remaining $\{x^i,\xi^j\}\neq 0$ for any value of $\omega\equiv\rho^2$.

We stress that the order of the matrix product (57) is not arbitrary but this is chosen in such a manner that x^0 and x^i correspond, in this order, to geodesic distances measured along time–like and space–like geodesics [25]. These quantities are called "geodesic parallel coordinates" and they can be interpreted as Cartesian coordinates on curved spacetimes. In particular, the metric on the (2+1)D spacetimes reads

$$\mathrm{d}\sigma^2 = \cosh^2(\rho x^1)\cosh^2(\rho x^2)(\mathrm{d}x^0)^2 - \cosh^2(\rho x^2)(\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2\,, \tag{60}$$

which under the limit $\rho\to 0$ reduces to the Minkowskian metric in flat Cartesian coordinates $\mathrm{d}\sigma^2 = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2$.

3.4 Non-commutative (anti)de Sitter spacetimes

The Poisson bracket $\{x^1,x^2\} = 0$ allows us to propose (but not to prove!) that the defining commutation relations of the (2+1)D non-commutative AdS and dS spacetimes are a direct Weyl quantization of the Poisson–Lie brackets (58), as no ordering problems appear in the commutation rules. By expanding them in power series in the curvature we find that

$$\begin{aligned}
&[\hat{x}^0, \hat{x}^1] = -z\,\frac{\tanh \rho\hat{x}^1}{\rho\cosh^2\rho\hat{x}^2} = -z\hat{x}^1 + \frac{z}{3}\,\omega(\hat{x}^1)^3 + z\omega\hat{x}^1(\hat{x}^2)^2 + o(\omega^2)\,,\\
&[\hat{x}^0, \hat{x}^2] = -z\,\frac{\tanh \rho\hat{x}^2}{\rho} = -z\hat{x}^2 + \frac{z}{3}\,\omega(\hat{x}^2)^3 + o(\omega^2)\,,\\
&[\hat{x}^1, \hat{x}^2] = 0\,.
\end{aligned} \tag{61}$$

Hence the linear relations in $\hat{x}^i$ correspond to the common "seed", κ–Minkowski, while corrections on the curvature start to arising in the third–order. Now three different quantum spacetimes are found; when $\omega \neq 0$ these can be interpreted as generalizations of κ–Minkowski with a non-zero cosmological constant. Notice that the "asymmetric" form of (61) could be expected from the beginning, as for instance the classical metric (60) shows. This is a consequence of dealing with intrinsic coordinates x^i of the spacetime. However it is possible to express (61) in a complete symmetric form by introducing non-commutative analogues of the 4D ambient space coordinates, where the (2+1)D spaces can be embedded [25].

To end with, we briefly comment on some properties of this deformation as well as on some open problems.

- Similarly to the previous τ (39), the deformation parameter z is a dimensionful one such that $[z] = [P_0]^{-1} = [\hat{x}^0] = 1/[\kappa]$, so that this can also be related with the Planck length l_P.
- Space isotropy is preserved at both the quantum algebra and group levels: $\hat{\theta}$ is absent from (61), the rotation J remains primitive in (48), and $\mathbf{P}, \mathbf{K}$ are transformed as classical vectors under J (49).
- Since the coproduct for the quantum $SO_\omega(2,2)$ group is still unknown, we have no quantum group transformations, under which, the quantum spacetimes should be coinvariant.
- The geometrical/physical role of the quantum coordinates $\hat{x}^i$ deserves a further study; their classical counterpart suggests that they may be interpreted as some kind of "geodesic operators".
- In κ–Poincaré with $\omega = 0$, the bracket (59) vanishes, so in the quantum case $[\hat{\xi}^i, \hat{\xi}^j] = 0$. This is consistent with the study developed in [57, 58] showing that κ–Poincaré boost transformations close a group and also that additivity of rapidity is preserved. Nevertheless, this is no longer true when a non-zero curvature is considered as $[\hat{\xi}^i, \hat{\xi}^j] \neq 0$, so that these properties may be either lost or somewhat modified for the quantum (anti)de Sitter algebras.

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