SU(1,1) algebra and interacting families of Calogero particles

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A one–dimensional model with interacting families of Calogero–type particles is studied. It includes harmonic, two–body and three–body interactions among particles. We find the exact eigenenergies corresponding to a class of the exact eigenstates of the model. We emphasize the universal $SU(1,1)$ structure of the model. We show how $SU(1,1)$ generators for the whole system are composed of $SU(1,1)$ generators of arbitrary subsystems. By imposing the conditions for the absence of the three–body interaction, we find certain relations between the coupling constants.

PACS: 03.65.Fd; 03.65.Sq
Key words: multispecies Calogero model, $SU(1,1)$ symmetry, interacting families

1 Introduction

More than thirty years ago, in his seminal paper [1] Calogero solved the 1D quantum–mechanical problem of N equal particles of mass m interacting pairwise with quadratic ('harmonic') and inverse–quadratic ('centrifugal') potentials.

The Hamiltonian of the model is essentially given by ($\hbar = 1$ and $i, j = 1, 2, \ldots, N$)

$$H = -\frac{1}{2m}\sum_i \frac{\partial_i^2}{\partial x_i^2} + \frac{m\omega^2}{2}\sum_i x_i^2 + \frac{\nu(\nu-1)}{2m}\sum_{i\neq j}\frac{1}{(x_i - x_j)^2}\,. \tag{1}$$

Here, $\nu(\nu-1) \geq -1/4$ is the dimensionless coupling constant and ω is the strength of a harmonic potential.

The exact wave functions of the Calogero model (1) are highly correlated. These correlations are encoded in the wave functions in the form of the Jastrow factor $\Delta = \prod_{i<j} |x_i - x_j|^\nu$. The exponent of the correlator is just the strength of the inverse–square interaction. For example, the ground state wave function is simply the product of the Jastrow factor and Gauss function

$$\Psi_0(x_1, x_2, \cdots, x_N) = \prod_{i<j} |x_i - x_j|^\nu \cdot \exp\left(-\frac{m\omega}{2}\sum_i x_i^2\right). \tag{2}$$

Calogero was also able to show that the complete set of energy eigenvalues had a form

$$E_{n_1,n_2,\cdots,n_N} = \omega\left(\frac{N}{2} + \frac{\nu N(N-1)}{2}\right) + \omega\sum_i n_i = E_0 + \omega\sum_i n_i\,. \tag{3}$$

The non-negative (integer) quantum numbers n_i obey selection rule $n_i \leq n_{i+1}$. It is remarkable that, apart from a constant shift for all energy levels, the spectrum (3) coincides with that of the N–body system with harmonic forces only. It may be also noted that the strength of the inverse–square interaction (ν) affects only the ground state energy E_0.

The Eq.(3) can be rearranged in the form of energy eigenvalues for free oscillators

$$E_{n_1,n_2,\cdots,n_N} = \omega\left(\frac{N}{2}\right) + \omega\sum_k \bar{n}_k \tag{4}$$

by introducing 'renormalized' quantum numbers $\bar{n}_k = n_k + \nu(k-1)$. It is clear that these $\bar{n}_k$s cannot be integers and they satisfy a new selection rule $\bar{n}_{k+1} - \bar{n}_k \geq \nu$, that is the lowest particle remains in its place, the second lowest one gets shifted by ν units etc. As a consequence, the Calogero model (1) provides a microscopic realization of the generalized Haldane exclusion statistics [2], with coupling constant ν playing the role of Haldane statistical parameter. In Haldane's formulation, however, there is the possibility of having particles of different species with a mutual statistical parameter depending on the species coupled. This suggests the generalization of the single-species 1D Calogero model to the multispecies 1D Calogero model. Distinguishability of the species can be introduced by allowing particles to have different masses (m_i) and different couplings (ν_{ij}) to each other. In fact, this kind of generalization has already been suggested in [1] (although not using statistical arguments) and since then several authors discussed it throughly [3] but in spite of a lot of efforts, up to now very little is known about spectra and wave functions of this model.

In the same reference [1], Calogero also considered a slightly generalized variant of the model (1), in which the N particles were partitioned into F families of N_a particles, $N = \sum N_a$, with the same harmonic interaction for all particles but with the inverse–square interaction acting only between particles belonging to the same family. The strength of the inverse–square interaction was allowed to be different for different families. It turns out that the spectrum of such a model is straightforward generalization of (3). The next step may be inclusion of interaction(s) between different families and some results for a such kind of model(s) are also available [4].

Recently, following our earlier investigation of a certain aspects of 1D Calogero model [5], we defined a multispecies Calogero model in 1D [6] and generalized it immediately to an arbitrary number of dimensions [7]. We succeeded in finding a class of exact eigenstates and eigenenergies of the both models. The analysis relied heavily on the $SU(1,1)$ algebraic structure of the corresponding Hamiltonians. Using the results obtained in Ref.[6], we were able to consider the problem of interacting families of Calogero–type particles in one dimension in a more transparent fashion than in [4] and in a more general way than in [1]. Precisely, we considered a model with a potential that generally includes harmonic, two–body and three–body interactions acting between particles belonging to different families, as well as the interaction between particles belonging to the same family with the coupling constant that may be different for different families [8].

In a present paper we review some salient features of the model studied in [6,8]. In Sec.2 we prepare all necessary tools for handling the problem of interacting families. We collect the main results of the analysis of the one–dimensional multispecies Calogero model [6]. We display the model Hamiltonian and discuss the conditions for the absence of the three–body interaction. Utilizing $SU(1,1)$ structure of the Hamiltonian, we reproduce a part of its spectrum corresponding to the global collective states. In Sec.3 we apply these results to the case of two interacting families of Calogero particles. We construct generators of $SU(1,1)$ algebra for interacting families and underline the importance of the dilatation part of the algebra, i.e. generator T_0. By imposing the conditions for the absence of the three–body interaction, we find certain relations between the coupling constants ('weak–strong' duality relations). The obvious generalization to the F–family model is briefly sketched in concluding Section 4.

2 1D multispecies Calogero model: main results

The model we are going to discuss is a multispecies generalization of (1). It is specified by masses of particles, m_i, and the coupling constants ω and ν_{ij}, $i,j = 1,2,\ldots,N$. The Hamiltonian reads:

$$H(\omega) = -\frac{1}{2}\sum_i \frac{1}{m_i}\frac{\partial^2}{\partial x_i^2} + \frac{\omega^2}{2}\sum_i m_i x_i^2 + \frac{1}{4}\sum_{i\neq j}\frac{\nu_{ij}(\nu_{ij}-1)}{(x_i-x_j)^2}\left(\frac{1}{m_i}+\frac{1}{m_j}\right) + \\ +\frac{1}{2}\sum_{i\neq j, i\neq k, j\neq k}\frac{\nu_{ij}\nu_{jk}}{m_j(x_j-x_i)(x_j-x_k)}\,. \tag{5}$$

The ground state wave function is of the Calogero type (2):

$$\Psi_0(x_1,\ldots,x_n) = \prod_{i<j}|x_i-x_j|^{\nu_{ij}}\cdot\exp\left(-\frac{\omega}{2}\sum_i m_i x_i^2\right) \equiv \\ \equiv \Delta\cdot\exp\left(-\frac{\omega}{2}\sum_i m_i x_i^2\right) \tag{6}$$

and the corresponding ground state energy is

$$E_0 = \omega\epsilon_0 = \omega\left(\frac{N}{2}+\sum_{i<j}\nu_{ij}\right). \tag{7}$$

A few additional remarks concerning the Hamiltonian (5) are in order. First, it describes distinguishable particles on the line, interacting with harmonic, two–body and three–body potentials. An earlier attempt to solve the similar, but less general Hamiltonian can be find in [3]. The appearance of the generalized Jastrow factor Δ in (6) has the same origin as in the Calogero model (1); namely, because the singular nature of the Hamiltonian (5) for $x_i = x_j$, the wave function (6) ought

to have a prefactor that vanish for coincident particles. Also, a stability condition demands that the two–body couplings $\nu_{ij}(\nu_{ij}-1)$ should be greater than $-1/4$, $\forall i,j$.

Let us consider the last term in Eq.(5), namely the three–body interaction, more closely. If we put $m_j = m = \text{const.}$ in (5) $(\forall j)$, symmetrize under the cyclic exchange of the indices $(i \to j \to k \to i)$ and reduce the sum to a common denominator using the identity

$$\sum_{\text{cycl.}} \frac{1}{(x_i - x_j)(x_i - x_k)} = 0\,,$$

we obtain that the necessary condition for vanishing of the three–body interaction is $\nu_{ij} = \nu = \text{const.}$ $(\forall i,j)$. In this way, the problem (5) is reduced to the ordinary N–body Calogero model with harmonic and two–body interactions only (1).

For the general ν_{ij} and m_j, it can be shown that the following conditions for the absence of the three–body term must hold:

$$\frac{\nu_{ij}\nu_{jk}}{m_j} = \frac{\nu_{jk}\nu_{ki}}{m_k} = \frac{\nu_{ki}\nu_{ij}}{m_i}\,, \quad \forall (i,j,k)\,. \tag{8}$$

The unique solution of these conditions is $\nu_{ij} = \lambda\, m_i\, m_j$, λ being some unspecified universal constant. It should be mentioned that the above arguments cannot be applied in higher dimensions, so the three-body interaction is generic for higher–dimensional multispecies model [7].

In an attempt to solve the Hamiltonian (5), we proceed in the following way. We extract the Jastrow factor Δ from the wave function (6) and perform a similarity transformation of the Hamiltonian (5), i.e.

$$\begin{aligned} \tilde{\Psi} &= \Delta^{-1}\Psi\,, \\ \tilde{H}(\omega) &= \Delta^{-1} H(\omega) \Delta\,. \end{aligned}$$

We find $\tilde{H}(\omega)$ as

$$\begin{aligned} \tilde{H}(\omega) &= -\frac{1}{2}\sum_i \frac{1}{m_i}\frac{\partial^2}{\partial x_i^2} + \frac{\omega^2}{2}\sum_i m_i x_i^2 - \frac{1}{2}\sum_{i\neq j} \frac{\nu_{ij}}{(x_i - x_j)}\left(\frac{1}{m_i}\frac{\partial}{\partial x_i} - \frac{1}{m_j}\frac{\partial}{\partial x_j}\right) = \\ &= \omega^2 T_+ - T_-\,. \end{aligned} \tag{9}$$

It should be stressed at this stage that the three–body interaction is not actually removed by similarity transformation (in a sense of preceding discussion); rather, it is hidden in $\tilde{H}(\omega)$. Nevertheless, the new Hamiltonian (9) has a much simpler structure than (5) and allows us to construct the representation of a spectrum generating algebra $SU(1,1)$. With help of its generators, we will be able to find algebraically a class of the exact eigenstates and eigenenergies of $\tilde{H}(\omega)$ and consequently, of $H(\omega)$.

We define the set of operators $\{T_\pm, T_0\}$ that satisfy the $SU(1,1)$ algebra

$$[T_-, T_+] = 2T_0\,, \quad [T_0, T_\pm] = \pm T_\pm\,.$$

as

$$T_- = -\tilde{H}(\omega = 0)\,, \quad T_+ = \frac{1}{2}\sum_i m_i x_i^2\,, \\ T_0 = \frac{1}{2}\left(\sum_i x_i \frac{\partial}{\partial x_i} + \epsilon_0\right). \tag{10}$$

Note that T_0 serves as a dilatation operator. One can deduce two important relations which will be extensively used later:

$$T_0\Delta = \left(\frac{1}{2}\sum_{i<j}\nu_{ij} + \frac{\epsilon_0}{2}\right)\Delta\,, \quad T_-\Delta = 0\,. \tag{11}$$

The Jastrow factor just appears to be the lowest weight vector for T_- and an eigenstate of dilatation part of the algebra.

The infinite set of exact eigenstates of the Hamiltonian (9) can be constructed by applying ladder operators

$$A_1^{\pm} = \frac{1}{\sqrt{2}}\left(\sqrt{M\omega}X \mp \frac{1}{\sqrt{M\omega}}\frac{\partial}{\partial X}\right) \tag{12}$$

and

$$B_2^{\pm} = \frac{1}{2}\left(\omega T_+ + \frac{T_-}{\omega}\right) \mp T_0 - \frac{1}{2}A_1^{\pm 2} \tag{13}$$

to the vacuum

$$\tilde{\Psi}_0(x_1, x_2, \ldots, x_N) = \tilde{\Psi}_0(X)\tilde{\Psi}_0(\xi_1, \xi_2, \ldots, \xi_N) = \\ = \exp\left(-\frac{M\omega}{2}X^2\right)\cdot\exp\left(-\frac{\omega}{2}\sum_i m_i\xi_i^2\right).$$

Here, we have introduced for convenience the center-of-mass (CM) coordinate $X = \frac{1}{M}\sum_i m_i x_i$ (where M is total mass of the system) and relative (R) coordinates $\xi_i = x_i - X$.

The exact eigenstates (corresponding to the center-of-mass states and global dilatation states) are

$$\tilde{\Psi}_{n_1 n_2} = \left(A_1^+\right)^{n_1}\left(B_2^+\right)^{n_2}\tilde{\Psi}_0\,, \quad n_1, n_2 = 0, 1, 2, \ldots\,. \tag{14}$$

Their eigenenergies can be deduced from relations

$$\tilde{H}_{\rm CM} = \frac{1}{2}\omega\{A_1^-, A_1^+\}_+\,, \quad \tilde{H}_R = \omega[B_2^-, B_2^+]\,, \\ [\tilde{H}_{CM}, A_1^{\pm}] = \pm A_1^{\pm}\,, \qquad [\tilde{H}_R, B_2^{\pm}] = \pm 2\omega B_2^{\pm}\,,$$

yielding

$$E_{n_1,n_2} = \omega(n_1 + 2n_2) + E_0\,. \tag{15}$$

The energy spectrum is linear in quantum numbers and highly degenerate. It can be shown that the dynamical symmetry responsible for this degeneracy is $SU(2)$, i.e. the same symmetry underlying the two–body Calogero model [6].

In the next section we apply these results to the case of two interacting families.

3 Two interacting families of Calogero particles

Let us consider two families, $\mathcal{F}_1$ and $\mathcal{F}_2$, of Calogero particles. The first one, denoted by $\mathcal{F}_1 = \{m_1, \nu_1, N_1\}$, is described by N_1 particles of mass m_1, the coupling constant ν_1 and the coordinates of the particles are $\{x_i\} = \{x_1, x_2, \ldots, x_{N_1}\}$. Similarly, the second one, denoted by $\mathcal{F}_2 = \{m_2, \nu_2, N_2\}$, is described by N_2 particles of mass m_2, the coupling constant ν_2 and the coordinates of the particles are $\{z_\alpha\} = \{z_1, z_2, \ldots, z_{N_2}\}$. The interaction strength between the first and the second family is $\nu_{12} = \kappa$.

The full Hamiltonian now reads

$$H(\omega) = H_1(\omega) + H_2(\omega) + H_{\rm int}\,, \tag{16}$$

where $H_{\rm int}$ is given by

$$\begin{aligned} H_{\rm int} = {} & \frac{1}{4}\sum_{i}^{N_1}\sum_{\alpha}^{N_2}\frac{\kappa(\kappa-1)}{(x_i-z_\alpha)^2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right) + \\ & + \frac{1}{4}\sum_{i}^{N_1}\sum_{\alpha\neq\beta}^{N_2}\left(\frac{\kappa^2}{m_1(x_i-z_\alpha)(x_i-z_\beta)}\right) + \frac{1}{2}\sum_{i}^{N_1}\sum_{\alpha\neq\beta}^{N_2}\left(\frac{\nu_2\kappa}{m_2(z_\alpha-x_i)(z_\alpha-z_\beta)}\right) + \\ & + \frac{1}{4}\sum_{i\neq j}^{N_1}\sum_{\alpha}^{N_2}\left(\frac{\kappa^2}{m_2(z_\alpha-x_i)(z_\alpha-x_j)}\right) + \frac{1}{2}\sum_{i\neq j}^{N_1}\sum_{\alpha}^{N_2}\left(\frac{\nu_1\kappa}{m_1(x_i-z_\alpha)(x_i-x_j)}\right), \end{aligned} \tag{17}$$

and $H_1(\omega)$ ($H_2(\omega)$) are generalized Calogero Hamiltonians, Eq.(5), for the first and the second family, respectively.

The corresponding ground state wave function of the Hamiltonian (17) is

$$\begin{aligned} \Psi_0(x_1,\ldots,x_{N_1},z_1,\ldots,z_{N_2}) = \prod_{i,\alpha}(x_i-z_\alpha)^\kappa \Psi_{0,1}(x_1,\ldots,x_{N_1})\Psi_{0,2}(z_1,\ldots,z_{N_2}) \equiv \\ \equiv \Delta_{12}\Psi_{0,1}(x_1,\ldots,x_{N_1})\Psi_{0,2}(z_1,\ldots,z_{N_2}) \equiv \Delta_{12}\Delta_1\Delta_2(\text{Gauss})\,, \end{aligned} \tag{18}$$

where $\Psi_{0,1}$ and $\Psi_{0,2}$ are the Calogero ground states (6) (when $\kappa = 0$) for the families $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively.

The ground state energy of the Hamiltonian (16) can be split into three terms:

$$\epsilon_0 = \epsilon_{0,1} + \epsilon_{0,2} + \kappa N_1 N_2\,, \tag{19}$$

describing the ground state energies of each family and the interaction between them.

We can perform a similarity transformation with a $\Delta_1\Delta_2$ part of the full Jastrow factor $\Delta = \Delta_1\Delta_2\Delta_{12}$ in (16, 18), to obtain

$$\begin{aligned} {\Delta_1}^{-1}{\Delta_2}^{-1} H(\omega)\Delta_1\Delta_2 &= \tilde{H}_1(\omega) + \tilde{H}_2(\omega) + H_{\rm int}\,, \\ {\Delta_1}^{-1}{\Delta_2}^{-1}\Psi_0 &= \prod_{i,\alpha}(x_i - z_\alpha)^\kappa \tilde{\Psi}_{0,1}\tilde{\Psi}_{0,2}\,. \end{aligned} \tag{20}$$

The $SU(1,1)$ structure for the interacting families can be read off as follows. Within each family, one can define $SU(1,1)$ generators $T_\pm^{(I)}$ and $T_0^{(I)}$ $(I = 1, 2)$ as in Eq.(10). These two sets of generators mutually commute. When interaction between families is switched on ($H_{\rm int} \neq 0$), the full set of the $SU(1,1)$ generators are

$$\begin{aligned} T_0 &= T_0^{(1)} + T_0^{(2)} + \tfrac{1}{2}\kappa N_1 N_2\,, \\ T_- &= T_-^{(1)} + T_-^{(2)} - H_{\rm int}\,, \\ T_+ &= T_+^{(1)} + T_+^{(2)}\,, \end{aligned} \tag{21}$$

and Eqs.(11) generalize to

$$T_0\Delta_{12} = \frac{1}{2}(\kappa N_1 N_2 + \epsilon_0)\Delta_{12}\,, \quad T_-\Delta_{12} = 0\,. \tag{22}$$

It is instructive to see what happens when the conditions (8) for the vanishing of the three–body interaction are satisfied. It means that the masses and coupling constant in our two–family system are related according to $\nu_{ij} = \lambda m_i m_j = \kappa$, or explicitly

$$\nu_1 = \lambda m_1^2\,, \quad \nu_2 = \lambda m_2^2\,, \quad \nu_{12} = \kappa = \lambda m_1 m_2\,, \tag{23}$$

from which it follows

$$\nu_1\nu_2 = \kappa^2 \;\Rightarrow\; \nu_2 = \left(\frac{m_2}{m_1}\right)^2 \nu_1 \tag{24}$$

Note that Eqs.(23, 24) imply that the couplings ν_1, ν_2 and κ have to be simultaneously positive, negative or zero.The connection between the coupling constants $\{\nu_1, \nu_2, \kappa\}$, Eq.(24), is sometimes ascribed to the 'weak–strong' coupling duality, but it is *de facto* a simple consequence of the absence of the three–body interaction in the starting Hamiltonian (5).

Let us demand that the ground state energies of the both families are equal, $\epsilon_{0,1} = \epsilon_{0,2}$, which implies (see Eqs.(3) or (7))

$$N_1 + \nu_1 N_1(N_1 - 1) = N_2 + \nu_2 N_2(N_2 - 1)\,. \tag{25}$$

If we fix $\kappa^2 = 1$ in (24), then the quadratic equation (25) has two distinct solutions

$$\begin{aligned} &\text{(i)} \quad \nu_1 = \frac{N_2 - 1}{N_1 - 1} > 0\,, \\ &\text{(ii)} \quad \nu_1 = -\frac{N_2}{N_1} < 0\,. \end{aligned}$$

Table 1. Physical implications

ν_1	κ	λ	ϵ_0	Comments
$\frac{N_2-1}{N_1-1}$	$+1$	$\lambda > 0$	$2N_1N_2 > 0$	Physical solution, no three–body interaction.
$-\frac{N_2}{N_1}$	$+1$	–	$N_1 + N_2 > 0$	Physical solution, with a three–body interaction.
$\frac{N_2-1}{N_1-1}$	-1	–	0	Unphysical solution, with a three-body interaction.
$-\frac{N_2}{N_1}$	-1	$\lambda < 0$	$N_1 + N_2 - 2N_1N_2 < 0$	Unphysical solution, no three–body interaction.

Their physical implications are summarized in Table 1.

In Refs.[5 – 7] we showed that there existed a critical point $\epsilon_{0R} = 0$ at which the system described by $\tilde{H}_R$ collapsed completely, i.e. the relative momenta, the relative energy and the relative coordinates were all zero at this critical point. The ground state was a square–integrable function only for $\epsilon_{0R} > 0$. This is the reason why we ascribe the term 'unphysical' to the last solutions in Table 1.

4 Conclusion and outlook

It is obvious how to generalize the results of the preceding section to the case of three and more families. For example, in the case of the three families of Calogero particles, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$, the wave function (18) contains Jastrow factors

$$\Delta = \Delta_1\Delta_2\Delta_3\Delta_{12}\Delta_{13}\Delta_{23}$$

and the composition law (21) generalizes to

$$\begin{aligned} T_0 &= \sum_{I=1}^{3} T_0^{(I)} + \frac{1}{2}\sum_{I<J} \nu_{IJ} N_I N_J \,, \\ T_- &= \sum_{I=1}^{3} T_-^{(I)} - H_{\rm int} \,, \\ T_+ &= \sum_{I=1}^{3} T_+^{(I)} \,. \end{aligned} \tag{26}$$

Similarly, Eqs.(11,22) now read

$$\begin{aligned} T_0\Big(\prod_{I<J}\Delta_{IJ}\Big) &= \frac{1}{2}\Big(\epsilon_0 + \sum_{I<J}\nu_{IJ}N_IN_J\Big)\Big(\prod_{I<J}\Delta_{IJ}\Big), \\ T_-\Big(\prod_{I<J}\Delta_{IJ}\Big) &= 0\,. \end{aligned} \tag{27}$$

For the general case of F–families the following relations, in connection with the composition law for the $SU(1,1)$ generators, should be used

$$\epsilon_0 = \sum_{I=1}^{F} \epsilon_{0,I} + \sum_{I<J;I,J=1}^{F} \epsilon_{0,IJ}\,,$$

$$\epsilon_{0,I} = \frac{N_I}{2} + \nu_I \frac{N_I(N_I-1)}{2}\,, \quad \epsilon_{0,IJ} = \nu_{IJ}\, N_I\, N_J\,.$$

We must emphasize that aforementioned relations between $SU(1,1)$ generators are universal for all choices of masses and coupling constants. Moreover, the same relations are valid for an arbitrary number of dimensions and for all potentials that behave as a kinetic energy term under the dilatation represented by the generator T_0. There is only one difference between one and higher dimensions. In the case of one dimension, as we already stated in connection with Eq.(8), one can exclude the three–body interaction between particles from the beginning, while there is no known way how to do this in dimensions higher than one. Our results can also be extended to other systems with the underlying conformal or superconformal symmetry.

Acknowledgments. One of the authors (M.M.) would like to thank Prof. Č. Burdik for kind invitation and hospitality during the XI International Conference on "Symmetry Methods in Physics", Prague, June 21 – 24, 2004.
This work was supported by the Ministry of Science and Technology of the Republic of Croatia under contracts No. 0098003 and No. 0119261.

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