

Newton–Wigner postulates and commutativity of position operators

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Commutativity of the position operator components is one of the Newton–Wigner postulates for the localized states tacitly included in the list. Omitting it gives additional possibilities to use the Poincare group representations for the analysis of the concept of the relativistic quantum localized states.

PACS: 02.20.Uw, 02.20.-a

Key words: localization, noncommutative, relativistic

1 Introduction. Motivations.

The present talk is devoted to an old problem of the particle position operator [2] in the relativistic quantum theory [1]. The notion of the relativistic coordinate is necessary element of any version of the quantum theory of particle interaction. First of all because we need to compare the measurement results with the theory predictions and consider the uncertainty relations. Even the old ideas on the space non-commutativity is also one of the approaches probing this key problem. We can understand this in a sense that in the relativistic quantum theory the adequate variables in terms of which the interaction is switched on in "natural" way are different from standard spatial coordinates and they are still expecting for their discovery.

In particular, the two body problem is important in this sense. In any case, if we have the bound state, we must be able to separate the motion of the center of mass and internal motion. The elementary systems [2] play important role as initial and final states of collision phenomena.

In the non-relativistic case the Galilean invariance of the system as a whole takes place in the case of potential functions $V(r)$ depending only on relative distance between interacting particles $r = |\mathbf{r}|$. In this case the coordinates of the center of mass \mathbf{R} and \mathbf{r} are separated. The free motion of the bound state (of the system as the whole) is described by the irreducible unitary representation of the Galilean group. From this point of view it would be natural to call the potentials of the type $V(r)$ the Galilean potentials. The internal motion of the system is reduced to the motion of the effective particle with the reduced mass μ in the field of potential. Even in the free case such separation is an important property. We also can call the two-body systems with the spherically symmetric potentials $V(r)$ the Galilean elementary systems by evident analogy with the relativistic particle localization

concept of E. Wigner [2].

To find the possibility to generalize these concepts to the relativistic case important challenge. Many physicists were occupied with search for its solution. Physically the situation is clear. At least the states like π -meson consist of the quark and anti-quark and their state vector describes the relativistic motion of the system as a whole. At least for the durations less than its life time. Note that according to E. Wigner [2] π -meson is an **elementary system**. Also we must not neglect the great effectiveness of the non-relativistic compound models for such systems. These non-relativistic models (with spherically symmetric potentials) work very efficiently, and they describe the bound states which are the elementary systems. But in fact these compound systems are relativistic and it is necessary to have the comprehensive relativistic potential model standing behind.

Also it is very important to stress that keeping in mind the potential relativistic models we seek for such relativistic analog of the relative coordinate \mathbf{r} on which the interaction potential depends so that the total relativistic invariance is respected in analogy with Galilean invariance of the non-relativistic two-particle problem with the potentials $V(r)$ depending on the relative distance between interacting particles.

This discussion can be continued but it is clear that old problem of finding the relativistic position operator still deserves to search for its solution. The basic ideas on this subject have been expressed by Newton and Wigner [2]. Their essential result is that for single particles a notion of the localizability and a corresponding **commuting** observables are uniquely determined by relativistic kinematics. On the other hand no relativistic quantum theory of interaction based on these ideas was constructed. In the present contribution we shall consider the possibility to introduce the concept of the **non-commuting** relativistic position operators obeying all Newton-Wigner postulates, having the transparent physical interpretation and admitting very simple quantum dynamical interpretation.

It must be stressed that the standard quantum-mechanical position operator $\hat{\mathbf{x}} = i\hbar\nabla_{\mathbf{p}}$ is connected with the Euclidean structures in terms of which the localization of a particle is considered. Let us quote here [2]: "Existence and uniqueness of a notion of localizability for a physical system are properties which depend only on the transformation law of the system under Euclidean group, i.e., the group of all space translations and rotations. The analysis of localizability in the Lorentz and Galilei invariant cases is then just a matter of discussing what representations of the Euclidean group can arise there". Both groups — Galilean and Poincaré contain the Euclidean group as their subgroup. But might be there are another realizations of the Euclidean group in the framework of the representation theory which allows another definition of the position operator? We try to show here that the answer is positive.

Now we recall two key notions of [2]. (There exists a huge literature on the subject, see for example [4] and references therein)

- **Elementary systems.** These are the systems whose states transform as an irreducible representation of the Poincaré group i.e. continuum wave func-

tions. This condition is quite unambiguous. "... All states of the system be obtainable from the relativistic transforms of *any* state by superpositions. In other words there must be no relativistically invariant distinction between the various states of the system which would allow for the principle of superposition" [2].

- **Particles.** According to E.Wigner these are structure-less objects which obey the restrictions: (a) Particle states form an elementary system. (b) "It should not be useful to consider the particle as a union of other particles". These words have been written more than 50 years ago. Despite the great progress in "particle physics", the clear notion of the particle still doesn't exist. The Wigner's notions of the elementary systems and particles is unfairly shadowed by another notions which doesn't make the situation more clear. We shall not go into the details of this theme and make only the hypothesis that bound states of elementary systems are realized in some cases as elementary systems. We shall call them the relativistic bound states.

The fact that the manifold of the physically realizable states contains only solutions with the positive energy has a number of consequences for the observables. Consider the solutions of the Klein–Gordon equation φ, ψ :

$$\varphi, \psi \in \{(+): p^\mu p_\mu = (p^0)^2 - \tilde{p}^2 = m^2 c^2, \quad p^0 \geq 0\}, \quad (1)$$

with the inner product

$$(\varphi, \psi) = \int_{(+)} d\Omega_{\mathbf{p}} \overline{\varphi(\mathbf{p})} \psi(\mathbf{p}), \quad d\Omega_{\mathbf{p}} = \frac{d\mathbf{p} m c}{p^0}. \quad (2)$$

The standard position operator

$$\hat{\mathbf{x}} = i\hbar \nabla_p \quad (3)$$

is non-hermitian in the metric (2):

$$\begin{aligned} (\varphi, \hat{\mathbf{x}}\psi) &= \int_{(+)} d\Omega_{\mathbf{p}} \overline{\varphi(\mathbf{p})} i\hbar \nabla_p \psi(\mathbf{p}) = \\ &= \int_{(+)} d\Omega_{\mathbf{p}} \overline{\left[\left(i\hbar \nabla_p - \frac{i\hbar \mathbf{p}}{\mathbf{p}^2 + m^2 c^2} \right) \varphi(\mathbf{p}) \right]} \psi(\mathbf{p}). \end{aligned} \quad (4)$$

So the operator $i\hbar \nabla_p$ does not correspond to any observable and can not be interpreted as an physical operator. It follows also that the Klein–Gordon wave function can not be considered as an probability amplitude to find the particle at the point \mathbf{x} at the moment of time x^0 .

To answer the question: what is the probability to find a particle at the point \mathbf{y} at some moment of time y^0 we must

1. Find the hermitian operator which can pretend to the role of the position;
2. Find its eigenfunctions $\psi_{\mathbf{y}, y^0}(x)$.

If particle is in the state with the wave function $\varphi(x)$, then the probability to find the particle at the point \mathbf{y} at the moment $y^0 = x^0$ will be $(\psi_{\mathbf{y},x^0}(x), \varphi)$.

The simplest way to obtain the position operator is to accept that the position operator is the hermitian part of $\hat{\mathbf{x}} = i\hbar\nabla_p$:

$$\hat{\mathbf{x}}_{\text{NW}} = \frac{1}{2} [\hat{\mathbf{x}} + \hat{\mathbf{x}}^\dagger] = i\hbar\nabla_p - \frac{i\hbar}{2} \frac{\mathbf{p}}{\mathbf{p}^2 + m^2c^2}. \quad (5)$$

Newton and Wigner derived this operator on a basis of a number of conditions which localized states must satisfy:

- (a) The states representing a system localized at time x^0 at \mathbf{y} must form a linear manifold S_0 , i.e. that the superposition of two such localized states be again localized in the same manner.
- (b) The set S_0 is invariant under rotations about the origin and reflections both of the spatial and of the time coordinate.
- (c) If a state ψ is localized (as above) at origin, a spatial displacement of ψ shall make it orthogonal to all states of S_0 .
- (d) Regularity condition: all generators of the Lorentz group preserve the normalizability condition: if ψ is normalizable then

$$\frac{(M^{\mu\nu}\psi, M^{\mu\nu}\psi)}{(\psi, \psi)} < \infty. \quad (6)$$

This condition excludes discontinuous states as localized ones and is essential in fixing the phase of the localized state wave function. In what follows we use the unit system in which $\hbar = c = 1$ if other is not stipulated.

Let the state $\psi_0(\mathbf{k})$ at the moment $x^0 = 0$ is localized at the origin. The translation operator in the **momentum space** is simply a factor

$$e^{-i\mathbf{k}\mathbf{a}},$$

so that the state $\psi_{\mathbf{y}}(\mathbf{k})$ obtained as the result of the translation of ψ_0 by \mathbf{y} is localized at \mathbf{y} at the moment $y^0 = x^0 = 0$ and has the form

$$\psi_{\mathbf{y}}(\mathbf{k}) = e^{-i\mathbf{k}\mathbf{a}}\psi_0(\mathbf{k}). \quad (7)$$

This transformed state, in accordance with (c) must be orthogonal to $\psi_0(\mathbf{k})$:

$$(\psi_{\mathbf{y}}, \psi_0) = \delta(\mathbf{y}) = \int d\Omega_{\mathbf{k}} |\psi_0(\mathbf{k})|^2 e^{-i\mathbf{k}\mathbf{a}} = \frac{1}{(2\pi)^3} \int d\mathbf{k} e^{-i\mathbf{k}\mathbf{a}}. \quad (8)$$

This is satisfied if $|\psi_0(\mathbf{k})|^2 = \frac{k^0}{(2\pi)^3}$. Taking into account the regularity condition (6) which forbids any \mathbf{k} -depending phase factor and putting the constant phase factor equal to 1 we obtain

$$\psi_0(\mathbf{k}) = (2\pi)^{-3/2} (k^0)^{1/2} \quad (9)$$

and for the wave function localized at $\mathbf{y} = 0$

$$\psi_{\mathbf{y}}(\mathbf{k}) = (2\pi)^{-3/2} e^{-i\mathbf{k}\mathbf{y}} (k^0)^{1/2}. \quad (10)$$

For the wave function of the localized state (at the moment x^0) $\psi_{\mathbf{y}}(\mathbf{x})$ in the configurational space we obtain

$$\begin{aligned} \psi_{\mathbf{y}}(\mathbf{x}) &= \int d\Omega_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \psi_{\mathbf{y}}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}}{\sqrt{k^0}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} = \\ &= \text{const} \left(\frac{mc}{\hbar r} \right)^{5/4} K_{5/4} \left(\frac{r}{\lambda_0} \right), \quad r = |\mathbf{x} - \mathbf{y}|, \quad \lambda_0 = \frac{\hbar}{mc}, \end{aligned} \quad (11)$$

$K_\nu(z)$ is MacDonald's function, λ_0 — Compton wave length of the particle.

Observations

First: $\psi_0(\mathbf{k})$ is the eigenfunction of the Newton–Wigner position operator:

$$\begin{aligned} \hat{\mathbf{x}}_{\text{NW}} \left\{ e^{-i\mathbf{k}\mathbf{y}} (k^0)^{1/2} \right\} &= i \left\{ \nabla_{\mathbf{k}} - \frac{1}{2} \frac{\mathbf{k}}{(k_0)^2} \right\} \left\{ e^{-i\mathbf{k}\mathbf{y}} k_0^{1/2} \right\} = \\ &= \mathbf{y} \left\{ e^{-i\mathbf{k}\mathbf{y}} k_0^{1/2} \right\}. \end{aligned} \quad (12)$$

Second: In the configurational space $\hat{\mathbf{x}}_{\text{NW}}$ is a non-local operator

$$\hat{\mathbf{x}}_{\text{NW}} = \left\{ \hat{\mathbf{x}} + \frac{1}{2} \frac{1}{\Delta + m^2 c^2} \nabla \right\}. \quad (13)$$

Third: The localized eigenfunction is not $\delta(\mathbf{x} - \mathbf{y})$ as in the non-relativistic theory, it is a function smeared in the spatial region of the size of the Compton wave length of the particle λ_0 , because $\delta(\mathbf{x} - \mathbf{y})$ can't be constructed from the positive frequency solutions only.

Fourth: The following commutation relations are satisfied:

$$\left[\hat{x}_{\text{NW}}^i, \hat{x}_{\text{NW}}^j \right] = 0, \quad \left[\hat{x}_{\text{NW}}^i, p^j \right] = i\delta_{ij}. \quad (14)$$

Fifth: $\hat{\mathbf{x}}_{\text{NW}}$ is a vector in respect to rotations. Under spatial translations

$$T_{\mathbf{a}} \hat{\mathbf{x}}_{\text{NW}} T_{\mathbf{a}}^{-1}. \quad (15)$$

Sixth: Non-relativistic limit

$$\hat{\mathbf{x}}_{\text{NW}} \longrightarrow \hat{\mathbf{x}} = i\nabla_{\mathbf{p}}. \quad (16)$$

The content of this article is as follows. In Sect.2 we consider the noncommutative alternative to the Newton-Wigner coordinate and related concept of the relativistic configurational space. In Sect.3 the two-body problem in the relativistic configurational space is formulated. In Sect.4 zero mass case is considered.

2 Alternative to Newton–Wigner approach

The Newton–Wigner theory uses essentially the momentum space. To establish the nonlocal operator $\hat{\mathbf{x}}_{\text{NW}}$ (13) in the configurational space would be very difficult.

But there is another circumstance essential for formulating the main idea of the present paper. In [1] the wave functions localized at different points are connected by translation:

$$\mathbf{x} \longrightarrow \mathbf{x} + \mathbf{a}, \quad e^{i\mathbf{k}(\mathbf{x}+\mathbf{a})} = e^{i\mathbf{k}\mathbf{x}}e^{i\mathbf{k}\mathbf{a}}. \quad (17)$$

The second relation has two mathematical meanings.

1. We consider, as in [1] the translations in the configurational space. Then the plane waves (exponentials) are the matrix elements of the irreducible unitary representations of the translation group numbered by the value of momentum \mathbf{k} . Fourier transformation is the expansion in matrix elements of the unitary irreducible representations of the translation group of the configurational space.
2. We consider (in contrast to [1]) the translations in the momentum \mathbf{k} -space. Then the same formula (17) describes the matrix element of the product of two irreps numbered by \mathbf{x} and \mathbf{a} correspondingly by the vector (of the momentum space) \mathbf{k} . The **inverse** Fourier transformation is the expansion in matrix elements of the unitary irreducible representations of the translation group of the momentum space space.

Such a symmetry between transformation within the same representation and the product of the representations is specific to the Euclidean translations. In the non-relativistic theory the difference between 1. and 2. is formal and unimportant because the geometries of the configurational and momentum spaces are isomorphic (mathematically) and Euclidean. Physical sense of the configurational and momentum spaces is different of course. The translations of the momentum space corresponds to Galilean transformations:

$$\begin{aligned} \mathbf{x} &\longrightarrow \mathbf{x} + \mathbf{V}t, \\ \dot{\mathbf{x}} &\longrightarrow \dot{\mathbf{x}} + \mathbf{V}, \\ m\dot{\mathbf{x}} &\longrightarrow m\dot{\mathbf{x}} + m\mathbf{V} \\ \mathbf{p} &\longrightarrow \mathbf{p} + \mathbf{k}, \quad \mathbf{p} = m\dot{\mathbf{x}}, \quad \mathbf{k} = m\mathbf{V}. \end{aligned} \quad (18)$$

The position operator (4) is the generator of translations of the momentum space.

Now we formulate the alternative to the Newton–Wigner concept. It is based on the simple observations.

1. From (4) we conclude that the geometry of the momentum space i.e. the manifold of realizable states of the relativistic particle of the positive frequency is the Lobachevsky space (1). We shall develop the one particle relativistic theory accepting this as the triggering point. Then we must substitute;

2. Galilean group \longrightarrow Lorentz group;
3. Galilean boosts \longrightarrow Lorentz boosts

$$\hat{\mathbf{x}} = i\hbar\nabla_p \longrightarrow \hat{\mathbf{x}}_{\text{rel}}, \quad (19)$$

where

$$\hat{\mathbf{x}}_{\text{rel}} = i\hbar\sqrt{1 + \frac{\mathbf{p}^2}{m^2c^2}} \nabla_p. \quad (20)$$

Thereby we consider (20) as the candidates for the relativistic position operators. But these are non-commuting operators. So proceeding along this geometrically natural way first what we must do is to give up with the commutativity of components of the position operator. We stress that in [1] the commutativity requirement is tacitly contained in the list of basic natural requirements.

First of all we note immediately that operators (20) are hermitian with the norm (2). After lifting the commutativity condition we can say that the new position operators are (the simplest!) hermitian operators in this metric.

Next we consider the problem of measurement. Evidently, in contrast with the commutative case the components of the position operator can not be measured. We note that A.Wightman refuses entirely to consider the non-commuting position operators from this point of view: "... operators which could not serve as position *observables* since their three components do not commute". But there are no reasons to reject the idea that the very concept of the configurational space in the relativistic case is modified as compared with the non-relativistic theory. As the consequence of such a modification must be the change of the all concept of the measuring the position, uncertainty relations etc.

To make this statements more clear, let us return for a time being to the non-relativistic case. As coordinates commute, we can diagonalize simultaneously all three components of it.

At the same time many other operators of the universal enveloping algebra of the Euclidean Lie algebra also are diagonal. For example the Casimir operator $\hat{\mathbf{x}}^2$ which is invariant operator of the Euclidean group of the *momentum space*

$$\begin{aligned} [\hat{\mathbf{x}}^2] e^{i\mathbf{p}\mathbf{x}} &= \Delta_{\mathbf{p}} e^{i\mathbf{p}\mathbf{x}} = \mathbf{x}^2 e^{i\mathbf{p}\mathbf{x}}, \\ \hat{x}^i e^{i\mathbf{p}\mathbf{x}} &= x^i e^{i\mathbf{p}\mathbf{x}}, \quad 0 \leq x < \infty, \quad -\infty < x^i < \infty \end{aligned} \quad (21)$$

Important is that the common eigenfunctions of these operators $e^{i\mathbf{p}\mathbf{x}}$ are the kernels of the Fourier transform connecting the Euclidean momentum space of the non-relativistic quantum mechanics and *corresponding configurational space*.

In the relativistic case it is natural to consider as the momentum space adequate from the physical point of view the space given by (1), i.e. the Lobachevsky space of the physical solutions of the Klein–Gordon equation. Integration over this space (with the Lorentz-invariant volume element $|D\Omega_{\mathbf{p}}|$ is given by (2). If we wish to follow the concept presented in the previous paragraph we should consider the universal enveloping algebra of the Lorentz group., determine the maximal set of

mutually commuting operators, determine their common eigenfunctions (new plane waves) and spectrum. The Casimir operator of the Lorentz group Lie algebra can be chosen in the form

$$\hat{r}^2 = \hat{\mathbf{x}}_{\text{rel}}^2 - \frac{\mathbf{M}^2}{m^2 c^2} - \frac{\hbar^2}{m^2 c^2}, \quad (22)$$

where \mathbf{M} is the angular momentum operator. The non-relativistic limit of (22) is $\hat{\mathbf{x}}^2$ (see (21)). Spectrum of r for the unitary representations takes continuous and discrete values. All these representations find the applications in various models of relativistic interactions. We shall concentrate on the so called principal series for which $0 \leq r < \infty$.

The eigenfunctions of \hat{r}^2 are the matrix elements of unitary irreducible representations of the Lorentz group or their generating functions — kernels of Gelfand–Graev transformations:

$$\hat{r}^2 \langle \mathbf{p} | \mathbf{r} \rangle = r^2 \langle \mathbf{p} | \mathbf{r} \rangle, \quad \langle \mathbf{r} | \mathbf{p} \rangle = \langle \mathbf{p} | \mathbf{r} \rangle^*. \quad (23)$$

They play the role of plane waves in the given relativistic formalism. Explicitly

$$\langle \mathbf{r} | \mathbf{p} \rangle = \left(\frac{p^0 - \mathbf{p} \mathbf{n}}{mc} \right)^{-1 - irmc/\hbar}, \quad \mathbf{n}^2 = 1. \quad (24)$$

The unit vector \mathbf{n} gives the sense to the symbol \mathbf{r} — by definition

$$\mathbf{r} = r \mathbf{n}. \quad (25)$$

We shall call the space of vectors \mathbf{r} the relativistic configurational space¹⁾. The partial expansion for the the plane wave (24) is

$$\begin{aligned} \langle \mathbf{r} | \mathbf{p} \rangle &= \sum_{l=0}^{\infty} i^l (2l+1) p_l(\cosh \chi, r) P_l(\mathbf{n}_p \cdot \mathbf{n}), \\ p^0 &= \cosh \chi, \quad \mathbf{p} = \sinh \chi \mathbf{n}_p, \quad \mathbf{n}_p^2 = 1, \end{aligned} \quad (26)$$

where

$$p_l(\cosh \chi, r) = (-1)^l \sqrt{\frac{\pi}{2 \sinh \chi}} \frac{\Gamma(ir + l + 1)}{\Gamma(ir + 1)} P_{-1/2+ir}^{-1/2+ir} \cosh \chi. \quad (27)$$

The expansion (26) is analogous to the non-relativistic one

$$e^{i\mathbf{p}\mathbf{r}} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(pr) P_l(\mathbf{n}_p \cdot \mathbf{n}), \quad (28)$$

where $j_l(pr) = \sqrt{\frac{\pi}{2pr}} J_{l+1/2}$ are the spherical Bessel functions. In the non-relativistic limit

$$p_l(\cosh \chi, r) \longrightarrow j_l(pr). \quad (29)$$

¹⁾ The concept of the relativistic configurational space have been introduced in [6] (see also [5]), for further references see [7]-[9]

The orthogonality and completeness conditions for the relativistic plane waves are

$$\begin{aligned} \frac{1}{(2\pi)^3} \int \langle \mathbf{r} | \mathbf{p} \rangle \langle \mathbf{p} | \mathbf{r}' \rangle d\Omega_{\mathbf{p}} &= \delta(\mathbf{r} - \mathbf{r}'), \\ \frac{1}{(2\pi)^3} \int \langle \mathbf{p} | \mathbf{r} \rangle \langle \mathbf{r} | \mathbf{p}' \rangle d\mathbf{r} &= \delta(\mathbf{p} - \mathbf{p}') = \delta(\mathbf{p} - \mathbf{p}') \frac{p^0}{mc}. \end{aligned} \quad (30)$$

The relativistic configurational space is an example of the quantum 3-dimensional Euclidean space. The quantum nature of the \mathbf{r} -space is predefined by the fact that the the Lie algebra of its isometry group is realized in a framework of noncommutative differential calculus. The momentum operators (generators of translations) are

$$\begin{aligned} H_0 &= \hat{p}^0 = \cosh\left(i \frac{\partial}{\partial r}\right) + \frac{i}{r} \sinh\left(i \frac{\partial}{\partial r}\right) - \frac{\Delta_{\vartheta, \psi}}{2r^2} \exp\left(i \frac{\partial}{\partial r}\right), \\ \hat{p}^1 &= -\sin \vartheta \cos \psi \left[\exp\left(i \frac{\partial}{\partial r}\right) - H_0 \right] - i \left(\frac{\cos \vartheta \cos \psi}{r} \frac{\partial}{\partial \vartheta} - \frac{\sin \psi}{r \sin \vartheta} \frac{\partial}{\partial \psi} \right) \exp\left(i \frac{\partial}{\partial r}\right), \\ \hat{p}^2 &= -\sin \vartheta \sin \psi \left[\exp\left(i \frac{\partial}{\partial r}\right) - H_0 \right] - i \left(\frac{\cos \vartheta \sin \psi}{r} \frac{\partial}{\partial \vartheta} + \frac{\cos \psi}{r \sin \vartheta} \frac{\partial}{\partial \psi} \right) \exp\left(i \frac{\partial}{\partial r}\right), \\ \hat{p}^3 &= -\cos \vartheta \left[\exp\left(i \frac{\partial}{\partial r}\right) - H_0 \right] + i \frac{\sin \vartheta}{r} \frac{\partial}{\partial \vartheta} \exp\left(i \frac{\partial}{\partial r}\right). \end{aligned} \quad (31)$$

They play the role of inner derivatives in relevant differential calculi. These operators mutually commute

$$[\hat{p}^\mu, \hat{p}^\nu] = 0, \quad \mu, \nu = 0, 1, 2, 3. \quad (32)$$

But the corresponding differentials of the coordinate functions don't commute with the coordinate functions themselves. For the details we refer the reader to [9] - [11]. Note that that the integration in the second formula in (30) is carried over with the Euclidean volume element $d\mathbf{r}$.

The common eigenfunctions of \hat{p}^μ are $\langle \mathbf{r} | \mathbf{p} \rangle$ (23)

$$\hat{p}^\mu \langle \mathbf{r} | \mathbf{p} \rangle = p^\mu \langle \mathbf{r} | \mathbf{p} \rangle, \quad (33)$$

from which we conclude that the "plane waves" (23) indeed describe the free relativistic motion with definite value of the 4-momentum. This is a new realization of the Lie algebra of the Euclidean group which we discussed in the **Introduction**.

Operators \hat{p}^μ identically satisfy the relativistic relation between energy and momentum (1). Important is also to note that these operators solve the problem of "extracting the root square" in the relation $\hat{p}^\mu = \sqrt{\mathbf{p}^2 + m^2 c^2}$:

$$\hat{p}^0 \langle \mathbf{r} | \mathbf{p} \rangle = p^0 \langle \mathbf{r} | \mathbf{p} \rangle = \sqrt{\mathbf{p}^2 + m^2 c^2} \langle \mathbf{r} | \mathbf{p} \rangle. \quad (34)$$

In the non-relativistic limit

$$|\mathbf{p}| \ll mc, \quad p^0 \cong mc + \frac{\mathbf{p}^2}{2mc}, \quad r \gg \frac{\hbar}{mc} \quad (35)$$

relativistic plane waves $\langle \mathbf{r} | \mathbf{p} \rangle$ transfer to usual plane waves

$$\begin{aligned} \langle \mathbf{r} | \mathbf{p} \rangle &= \exp \left[- \left(1 + i r \frac{mc}{\hbar} \right) \ln \left(\frac{p^0 - \mathbf{p}\mathbf{n}}{mc} \right) \right] \cong \\ &\cong \exp \left[- \left(1 + i r \frac{mc}{\hbar} \right) \ln \left(1 - \frac{\mathbf{p}\mathbf{n}}{mc} + \frac{\mathbf{p}^2}{2m^2c^2} + \dots \right) \right] \cong \\ &\cong \exp \left(i \frac{\mathbf{p} \cdot (\mathbf{r}\mathbf{n})}{\hbar} \right) = \exp \left(i \frac{\mathbf{p}\mathbf{r}}{\hbar} \right). \end{aligned} \quad (36)$$

The wave function of the particle can be expanded in the Fourier integral in the relativistic plane waves

$$\psi(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int \langle \mathbf{r} | \mathbf{p} \rangle \psi(\mathbf{p}) d\Omega_{\mathbf{p}}. \quad (37)$$

Particles are localized in the relativistic configurational space in a usual sense. The position operator $\hat{\mathbf{r}}$ in \mathbf{r} -representation acts on a wave function in a usual way

$$\hat{\mathbf{r}}\psi(\mathbf{r}) = \mathbf{r}\psi(\mathbf{r}). \quad (38)$$

The eigenfunctions $\psi_{\mathbf{r}_0}(\mathbf{r})$ of $\hat{\mathbf{r}}$ corresponding to the eigenvalue \mathbf{r}_0 are $\psi_{\mathbf{r}_0}(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_0)$ so that

$$\hat{\mathbf{r}}\psi_{\mathbf{r}_0}(\mathbf{r}) = \mathbf{r}\psi_{\mathbf{r}_0}(\mathbf{r}). \quad (39)$$

Eigenfunctions corresponding to different eigenvalues — i.e. the states localized at different points \mathbf{r}_0 and $\tilde{\mathbf{r}}_0$ are orthogonal

$$\int \psi_{\mathbf{r}_0} \psi_{\tilde{\mathbf{r}}_0} d\mathbf{r} = \delta(\tilde{\mathbf{r}}_0 - \mathbf{r}_0), \quad (40)$$

which is the usual localization condition in the new relativistic configurational space.

3 Relativistic two-body problem.

We start with the non-relativistic two-body problem. In the non-relativistic theory the two-body Hamiltonian operator

$$H = -\frac{\hbar^2}{2m_1} \Delta_{r_1} - \frac{\hbar^2}{2m_2} \Delta_{r_2} + V(r), \quad (41)$$

$$H = -\frac{\hbar^2}{2M} \Delta_R - \frac{\hbar^2}{2\mu} l\Delta_r + V(r), \quad (42)$$

where

$$M = m_1 + m_2, \quad \mu = \frac{m_1 m_2}{M}, \quad \mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{M}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \quad (43)$$

or in the momentum space variables

$$H = \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}^2}{2\mu} + V(r), \quad (44)$$

where

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad \mathbf{p} = \frac{\mathbf{p}_1}{m_1} - \frac{\mathbf{p}_2}{m_2}. \quad (45)$$

The way we consider here the two–body problem is rather artificial but it admits the natural generalization to the relativistic two–body problem in the framework of the approach considered here. Consider the free motion wave functions of the separate particles $\psi_1^{(0)}(\mathbf{r}_1)$, $\psi_2^{(0)}(r_2)$. Writing them in terms of the CM and relative coordinates \mathbf{R} and \mathbf{r} we obtain the ”bilocal” dependence of the individual wave functions

$$\begin{aligned} \psi_1^{(0)}(\mathbf{r}_1) &= e^{i\mathbf{p}_1\mathbf{r}_1} = \psi_1^{(0)}(\mathbf{R}, \mathbf{r}) = e^{i\mathbf{p}_1\mathbf{R}} e^{i\mathbf{p}_1 \frac{m_2}{M} \mathbf{r}}, \\ \psi_2^{(0)}(\mathbf{r}_2) &= e^{i\mathbf{p}_2\mathbf{r}_2} = \psi_2^{(0)}(\mathbf{R}, \mathbf{r}) = e^{i\mathbf{p}_2\mathbf{R}} e^{-i\mathbf{p}_2 \frac{m_1}{M} \mathbf{r}}. \end{aligned} \quad (46)$$

This reflects the simple fact that these wave functions can be obtained from the individual wave functions of the separate particles in the CM system (i.e. $\exp\left(i\mathbf{p}_1 \frac{m_2}{M} \mathbf{r}\right)$, and $\exp\left(-i\mathbf{p}_2 \frac{m_1}{M} \mathbf{r}\right)$ where $\frac{m_2}{M} \mathbf{r}$, and $-\frac{m_1}{M} \mathbf{r}$ are the coordinates of the first and the second particle in the CM system correspondingly) by the translation by \mathbf{R} . In this way we factorize the dependence on the CM coordinate \mathbf{R} , and relative coordinate \mathbf{r} for individual particles. Now taking the products of the corresponding parts of these wave functions, we restore the CM coordinate \mathbf{R} dependence

$$\Phi_{\text{CM}}(\mathbf{R}) = e^{i\mathbf{p}_1\mathbf{R}} e^{i\mathbf{p}_2\mathbf{R}} = e^{i\mathbf{P}\mathbf{R}} \quad (47)$$

and relative coordinate \mathbf{r} dependence (effective particle free motion)

$$\phi_{\text{eff}}^{(0)}(\mathbf{r}) = \exp\left(i\mathbf{p}_1 \frac{m_2}{M} \mathbf{r}\right) \exp\left(-i\mathbf{p}_2 \frac{m_1}{M} \mathbf{r}\right) = \exp\left[i\left(\frac{\mathbf{p}_1}{m_1} - \frac{\mathbf{p}_2}{m_2}\right)\mu\mathbf{r}\right] = e^{i\mathbf{p}\mathbf{r}} \quad (48)$$

of the two–body wave function

$$\psi^{(0)}(\mathbf{r}_1, \mathbf{r}_2) = \psi^{(0)}\left(\mathbf{R} + \frac{m_2}{M}\mathbf{r}, \mathbf{R} - \frac{m_1}{M}\mathbf{r}\right) = T_{\mathbf{R}}\psi^{(0)}\left(\frac{m_2}{M}\mathbf{r}, -\frac{m_1}{M}\mathbf{r}\right), \quad (49)$$

$$T_{\mathbf{R}}\mathbf{r}_i = \mathbf{r}_i + \mathbf{R}, \quad (50)$$

$$\psi^{(0)}(\mathbf{r}_1, \mathbf{r}_2) = \psi_1^{(0)}(\mathbf{r}_1)\psi_2^{(0)}(\mathbf{r}_2) = \phi^{(0)}(\mathbf{R}, \mathbf{r}) = E^{i\mathbf{P}\mathbf{R}} e^{i\mathbf{p}\mathbf{r}}. \quad (51)$$

So that

$$\phi^{(0)}(\mathbf{R}, \mathbf{r}) = \Phi_{\text{CM}}(\mathbf{R})\phi_{\text{eff}}^{(0)}(\mathbf{r}) \quad \text{and} \quad \phi_{\text{eff}}^{(0)}(\mathbf{r}) = \phi^{(0)}(\mathbf{0}, \mathbf{r}). \quad (52)$$

In the presence of the interaction $\phi_{\text{eff}}^{(0)}(\mathbf{r})$ is changed for $\phi_{\text{eff}}(\mathbf{r})$ which is not already factorized and satisfies the Schrödinger equation

$$\left[\frac{\mathbf{p}^2}{2\mu} + V(r)\right]\phi_{\text{eff}}(\mathbf{r}) = E\phi_{\text{eff}}(\mathbf{r}). \quad (53)$$

The effective wave function $\phi_{\text{eff}}(\mathbf{r})$ can be expressed in terms of the corresponding wave function in the momentum space

$$\phi_{\text{eff}}(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int e^{i\mathbf{p}\mathbf{r}} \phi_{\text{eff}}(\mathbf{p}) d\mathbf{p}. \quad (54)$$

The system as a whole moves with the constant momentum, so that $\Phi_{\text{cm}}(\mathbf{R})$ does not change in the interacting case. The factorization of the individual wave functions does not take place but the relation (52) is valid so the Fourier transform for the total wave function $\phi(\mathbf{R}, \mathbf{r}) = \Phi_{\text{cm}}(\mathbf{R})\phi_{\text{eff}}(\mathbf{r})$ is

$$\phi(\mathbf{R}, \mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int e^{i\mathbf{K}\mathbf{R}} e^{i\mathbf{k}\mathbf{r}} \phi_{\mathbf{P}}(\mathbf{K}, \mathbf{k}) d\mathbf{K} d\mathbf{k}, \quad (55)$$

where

$$\phi_{\mathbf{P}}(\mathbf{K}, \mathbf{k}) = \delta(\mathbf{P} - \mathbf{K}) \phi_{\text{eff}}(\mathbf{p}) \quad (56)$$

In the relativistic configurational \mathbf{r} -space no local addition theorem like (17) exists and we must use the expansion (26). From this expansion the following "nonlocal" addition theorem follows [6]

$$\int \langle \mathbf{r} | \mathbf{p}_1 \rangle \langle \mathbf{p}_2 | \mathbf{r} \rangle d\mathbf{n} = \int \langle \mathbf{r} | \mathbf{p}_1(-) \mathbf{p}_2 \rangle d\mathbf{n}, \quad (57)$$

where $\mathbf{q} = \mathbf{p}_1(-)\mathbf{p}_2$ is a vector \mathbf{p}_1 boosted into the Lorentz frame moving with the velocity $\mathbf{v} = \frac{\mathbf{p}_2 c}{\sqrt{\mathbf{p}_2^2 + m^2 c^2}}$

$$\begin{aligned} \mathbf{q} &= \mathbf{p}_1(-)\mathbf{p}_2, \\ (\mathbf{p}_1(-)\mathbf{p}_2)_0 &= (\cosh \chi_1 \cosh \chi_2 - \sinh \chi_1 \sinh \chi_2 (\mathbf{n}_{p_1} \cdot \mathbf{n}_{p_2})). \end{aligned} \quad (58)$$

Of course for the standard plane waves the integral addition theorem like (57) is valid

$$\int \exp\left(i\mathbf{p}_1 \frac{m_2}{M} \mathbf{r}\right) \cdot \exp\left(-i\mathbf{p}_2 \frac{m_1}{M} \mathbf{r}\right) d\mathbf{n} = \int e^{i\mathbf{p}\mathbf{r}} d\mathbf{n}. \quad (59)$$

The formula (59) becomes necessary if we wish to multiply not the exponents themselves but their partial expansions (28). We see that the integration over $d\mathbf{n}$ is necessary to restore the partial expansion

$$e^{i\mathbf{p}\mathbf{r}} = \sum_{l=0}^{\infty} i^l (2l+1) j_l \left(\left| \frac{m_2 \mathbf{p}_1 - m_1 \mathbf{p}_2}{M} \right| \right) P_l(\mathbf{n}_p \cdot \mathbf{n}) \quad (60)$$

in the right hand side of (48). But such angular averaging commutes with the Galilean Hamiltonians (spherically symmetric potentials)

$$\int \left[\frac{\mathbf{p}^2}{2\mu} + V(r) \right] \psi(\mathbf{r}) d\mathbf{n} = \left[\frac{\mathbf{p}^2}{2\mu} + V(r) \right] \int \psi(\mathbf{r}) d\mathbf{n}. \quad (61)$$

Transferring to the relativistic two–body problem we must first note that hyperboloids in the momentum space corresponding to different particles are different, see (1). This must be taken into account for example in (24):

$$\langle \mathbf{r} | \mathbf{p}_i \rangle = (\cosh \chi_i - \sinh \chi_i (\mathbf{n}_{p_i} \cdot \mathbf{n}))^{-1 - i r m_i c / \hbar}, \quad i = 1, 2, \quad (62)$$

where r is the (dimensional) analog of the relative distance between particles in the relativistic configurational space.

Thus the free relativistic two body wave function $\phi_{\text{eff}}^{(r,0)}(\mathbf{r})$ by analogy with (48) can be chosen in a form

$$\phi_{\text{eff}}^{(r,0)}(\mathbf{r}) = \left\langle \frac{m_2}{M} \mathbf{r} \middle| \mathbf{p}_1 \right\rangle \left\langle \mathbf{p}_2 \middle| \frac{m_1}{M} \mathbf{r} \right\rangle. \quad (63)$$

It describes the free motion in the CM system, the relativistic free motion (see Sec.2), and has the right non-relativistic limit. There are several possibilities to generalize for the relativistic case the formula (48) but our choice is the simplest from the formal point of view and most transparent from the physical point of view. In explicit form

$$\begin{aligned} \phi_{\text{eff}}^{(r,0)}(\mathbf{r}) &= (\cosh \chi_1 - \sinh \chi_1 (\mathbf{n}_{p_1} \cdot \mathbf{n}))^{-1 - i(m_2/M)(r m_1 c / \hbar)} \times \\ &\times (\cosh \chi_2 - \sinh \chi_2 (\mathbf{n}_{p_2} \cdot \mathbf{n}))^{-1 + i(m_1/M)(r m_2 c / \hbar)}. \end{aligned} \quad (64)$$

Now we apply the addition theorem (57) and obtain

$$\int \left\langle \frac{m_2}{M} \mathbf{r} \middle| \mathbf{p}_1 \right\rangle \left\langle \mathbf{p}_2 \middle| \frac{m_1}{M} \mathbf{r} \right\rangle d\mathbf{n} = \int \langle \mathbf{r} | \mathbf{q} \rangle d\mathbf{n}, \quad (65)$$

where q is given by (58). Remarkable is that the mass entering the expression for the relativistic plane wave in the right hand side of (65) is the reduced mass (43)

$$\langle \mathbf{r} | \mathbf{q} \rangle = (\cosh \chi_q - \sinh \chi_q (\mathbf{n}_q \cdot \mathbf{n}))^{-1 - i r \mu c / \hbar}. \quad (66)$$

We shall consider (66) as the free relativistic effective wave function describing the relative motion. In the presence of potential the $\phi_{\text{eff}}^{(0)}(\mathbf{r})$ is modified and we have analogously to (54)

$$\phi_{\text{eff}}^{(r)}(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int \langle \mathbf{r} | \mathbf{k} \rangle \phi_{\text{eff}}^{(r)}(\mathbf{k}) d\Omega \mathbf{k}. \quad (67)$$

Now we consider the arbitrary frame of reference. In absence of the external field the our 2–body system moves with the constant velocity. This allows us to write the relativistic 2–body wave function in the form indistinguishable from (56)

$$\phi_{\mathbf{P}}^{(r)}(\mathbf{K}, \mathbf{k}) = \delta(\mathbf{P} - \mathbf{K}) \phi_{\text{eff}}^{(r)}(\mathbf{p}) \quad (68)$$

and

$$\begin{aligned}\phi_{\mathbf{P}}^{(r)}(\mathbf{R}, \mathbf{r}) &= T_{\mathbf{R}}\phi_{\text{eff}}^{(r)}(\mathbf{r}) = T_{\mathbf{R}} \frac{1}{(2\pi)^{3/2}} \int \langle \mathbf{r}|\mathbf{k} \rangle \phi_{\text{eff}}^{(r)}(\mathbf{k}) d\Omega \mathbf{k} = \\ &= \frac{1}{(2\pi)^{3/2}} \int e^{i\mathbf{P}\mathbf{R}} \langle \mathbf{r}|\mathbf{k} \rangle \phi_{\text{eff}}^{(r)}(\mathbf{k}) d\Omega \mathbf{k}\end{aligned}\quad (69)$$

or

$$\phi_{\mathbf{P}}^{(r)}(\mathbf{R}, \mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int e^{i\mathbf{K}\mathbf{R}} e^{i\mathbf{k}\mathbf{r}} \phi_{\mathbf{P}}^{(r)}(\mathbf{K}, \mathbf{k}) d\mathbf{K} d\Omega \mathbf{k}.\quad (70)$$

Now bilocal character of the 2-body wave function in contrast to the non-relativistic case becomes essential because the variables \mathbf{R} , and \mathbf{r} have the different nature.

4 Massless case

This section arose as the result of the discussion with Prof. J. Niederle during my presentation of this talk at the Symposium. See in this connection [3, 12]

In distinction with the case $m \neq 0$ the relativistic plane waves for the massless particles $\langle \rho, \alpha | s, \phi \rangle$ are singular functions and must be regularized [13]. As we shall see no other singularities emerge in the theory of the massless particles in the framework of the relativistic localization concept. Accepting the choice of the plane wave regularization once and forever we resolve the old problem of massless relativistic particle localization [3, 12]. We define the regularized massless plane waves as [14]

$$\langle \tilde{\rho} | \tilde{p} \rangle = \langle \rho, \alpha | s, \phi \rangle = s^{-i\rho-1/2} (1 - \cos(\phi - \alpha))_+^{-i\rho-1/2},\quad (71)$$

where the generalized function x_+ is defined as

$$x_+^\lambda = \begin{cases} 0 & x \leq 0 \\ x^\lambda & x > 0 \end{cases}\quad (72)$$

or

$$x_+^\lambda = \frac{e^{i\pi\lambda} (x - i\epsilon)^\lambda - e^{-i\pi\lambda} (x + i\epsilon)^\lambda}{2i \sin i\pi\lambda}.\quad (73)$$

The case of the massless scalar particle in two spatial dimensions is considered. The point on the upper pole of the cone

$$p^\mu p_\mu = (p^0)^2 - \tilde{p}^2 = 0, \quad p^0 \geq 0, \quad \tilde{p}^2 = p_1^2 + p_2^2\quad (74)$$

is parameterized as

$$\{p^\mu\} = \{s, s \cos \phi, s \sin \phi\}, \quad 0 \leq s < \infty, \quad 0 \leq \phi < 2\pi.\quad (75)$$

Correspondingly the relativistic configurational space is 2-dimensional. The coordinates in it are given as

$$\tilde{\rho} = \{\rho \cos \alpha, \rho \sin \alpha\}, \quad 0 \leq \rho < \infty, \quad 0 \leq \alpha < 2\pi.\quad (76)$$

The series expansion of the plane wave (71) has the form

$$\begin{aligned} \langle \rho, \alpha | s, \phi \rangle &= \frac{2^{-i\rho} \Gamma(1/2 - i\rho) \Gamma(-i\rho) s^{-i\rho - 1/2}}{\sqrt{2\pi}} \times \\ &\times \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{in(\phi - \alpha)}}{\Gamma(1/2 + n - i\rho) \Gamma(1/2 - n - i\rho)}. \end{aligned} \quad (77)$$

The plane waves obey the orthogonality

$$\frac{1}{(2\pi)^2} \int \langle \tilde{\rho} | \tilde{p} \rangle \langle \tilde{p} | \tilde{\rho}' \rangle d\Omega_p = \frac{\delta(\tilde{\rho} - \tilde{\rho}')}{\mu(\rho)}, \quad (78)$$

where

$$\mu(\rho) = \rho \tanh \pi \rho \quad (79)$$

and completeness conditions

$$\frac{1}{(2\pi)^2} \int \langle \tilde{p} | \tilde{\rho} \rangle \langle \tilde{\rho} | \tilde{p}' \rangle d\Omega_p = |\tilde{p}| \delta(\tilde{p} - \tilde{p}'), \quad (80)$$

where

$$d\Omega_p = \frac{d\tilde{p}}{|\tilde{p}|}, \quad d\Omega_\rho = \rho \tanh \pi \rho. \quad (81)$$

The localization problem in the relativistic configurational space for massless particles in principle does not differ from the case $m \neq 0$. Particles are localized in the relativistic configurational space in a usual sense. The position operator $\hat{\rho}$ in ρ -representation acts on a wave function in a usual way

$$\hat{\rho}\psi(\rho) = \rho\psi(\rho). \quad (82)$$

The eigenfunctions $\psi_{\rho_0}(\rho)$ of $\hat{\rho}$ corresponding to the eigenvalue ρ_0 are $\psi_{\rho_0}(\rho) = \delta(\rho - \rho_0)$ so that

$$\hat{\rho}\psi_{\rho_0}(\rho) = \rho_0\psi_{\rho_0}(\rho). \quad (83)$$

Eigenfunctions corresponding to different eigenvalues — i.e. the states localized at different points ρ_0 and $\tilde{\rho}_0$ are orthogonal

$$\int \psi_{\rho_0} \psi_{\tilde{\rho}_0}^* d\rho = \delta(\tilde{\rho}_0 - \rho_0), \quad (84)$$

which is the usual localization condition in the new relativistic configurational space.

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