Extension of Moyal−deformed hierarchies of soliton equations

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Moyal–deformed hierarchies of soliton equations can be extended to larger hierarchies by including additional evolution equations with respect to the deformation parameters. A general framework is presented in which the extension is universally determined and which applies to several deformed hierarchies, including the noncommutative KP, modified KP, and Toda lattice hierarchy. We prove a Birkhoff factorization relation for the extended ncKP and ncmKP hierarchies. Also reductions of the latter hierarchies are briefly discussed. Furthermore, some results concerning the extended ncKP hierarchy are recalled from previous work.

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1 Introduction

Classical soliton equations arise as compatibility conditions of linear systems (see [1, 2], for example). Generalizing the ordinary product of functions of two coordinates x, t to some associative (and not necessarily commutative) product $*$, for which the partial derivatives $\partial := \partial_x$ and ∂_t are derivations, let us take the linear system to be of the form

$$\psi_x = U * \psi \,, \quad \psi_t = V * \psi \,, \tag{1.1}$$

where U, V are matrices of functions or suitable operators, depending on some (matrix–valued) field $u(x,t)$. The compatibility condition then reads

$$U_t - V_x + [U,V]_* = 0 \,, \tag{1.2}$$

where an index x or t indicates a corresponding partial derivative, and $[U,V]_* := U*V - V*U$. If this equation is equivalent to a nonlinear partial differential equation (PDE) for the dependent variable u, the differential equation is said to admit a zero curvature formulation. If the latter is non-trivial, the PDE possesses certain properties typically associated with completely integrable models, in particular

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soliton equations. Generalizations of the underlying product of functions in the aforementioned sense have been treated in particular in [3, 4, 5, 6]. What happens if the product depends on the coordinates or additional parameters? An interesting example is given by the Moyal–product:

$$f * h := \mathbf{m}(\mathrm{e}^{P/2} f \otimes h) , \quad P := \theta(\partial_t \otimes \partial_x - \partial_x \otimes \partial_t) , \tag{1.3}$$

where θ is a parameter, $\mathbf{m}(f \otimes h) = fh$ with the ordinary product of the two functions f, h on the right hand side, and the tensor product has to be understood over $\mathbb{C}[[\theta]]$ (or with $\mathbb{C}$ replaced by another field). The new parameter θ may now be treated on an equal footing with the parameters t and x. It is then natural to ask for an extension of the above linear system by an additional equation of the form

$$\psi_\theta = W * \psi . \tag{1.4}$$

Of course, this leads to additional compatibility conditions and a θ–evolution equation for the field u. In the examples we have explored so far [7, 8], such an extension indeed exists, without restrictions imposed on the original linear system. This new structure will be the central object in the following. More precisely, we will concentrate on hierarchies, which arise from a linear system of the above form and consist of evolution equations with 'evolution times' t_n, $n \in \mathbb{N}$, the flows of which commute with each other (see section 2). In this case, a deformation parameter $\theta_{m,n}$ can be associated with each pair of parameters t_m, t_n. The above procedure with P in (1.3) replaced by

$$P := \sum_{m,n=1}^{\infty} \theta_{m,n} \, \partial_{t_m} \otimes \partial_{t_n} , \tag{1.5}$$

then results in an extension of this hierarchy by additional evolution equations with respect to the parameters $\theta_{m,n} = -\theta_{n,m}$. The new θ–evolution equations are non-autonomous equations.[1)] Fixing one equation of the extended hierarchy, all others are *symmetries* of this equation. The θ–evolution equations are thus new symmetries of the Moyal–deformed hierarchy equations.

Section 2 formulates an abstract scheme for the extension of Moyal–deformed Lax hierarchies and so-called N–reductions (see [2], for example) of the latter. Section 3 presents examples in the framework of Sato theory: the extensions of the Moyal–deformed (matrix) KP [7, 8] and mKP hierarchies, for which we also prove a Birkhoff factorization relation. Another example is treated in section 4: the extension of the Moyal–deformed Toda lattice hierarchy. Some remarks are collected in section 5.

2 Extension of Moyal−deformed Lax hierarchies

Let $(\mathcal{A}, *)$ be an associative algebra and $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ a decomposition of the Lie algebra $(\mathcal{A}, [\,,\,]_*)$, where $[\ ,\ ]_*$ is the commutator in the algebra $(\mathcal{A}, *)$, into Lie

[1)] Note that the classical KP hierarchy admits so-called 'additional symmetries' which are also non-autonomous since they depend explicitly on the variables t_n. See [2], for example.

subalgebras $\mathcal{A}_\pm$, so that

$$[\mathcal{A}_-, \mathcal{A}_-]_* \subset \mathcal{A}_- \,, \quad [\mathcal{A}_+, \mathcal{A}_+]_* \subset \mathcal{A}_+ \,. \tag{2.1}$$

Let $(\ )_+$ and $(\ )_-$ denote the projections to $\mathcal{A}_+$, respectively $\mathcal{A}_-$.

For $L \in \mathcal{A}$, depending on functions of variables t_n, $n \in \mathbb{N}$, consider the set of equations

$$L_{t_n} = [(L^n)_+, L]_* \equiv -[(L^n)_-, L]_* \,, \quad n = 1, 2, \dots \,, \tag{2.2}$$

where $L^n = L * L * \ldots * L$. This implies

$$(L^m)_{t_n} = [(L^n)_+, L^m]_* \,, \quad m = 1, 2, \dots \,. \tag{2.3}$$

Using the identity

$$[(L^n)_+, (L^m)_-]_* + [(L^n)_-, (L^m)_+]_* + [(L^n)_+, (L^m)_+]_* + [(L^n)_-, (L^m)_-]_* = 0 \tag{2.4}$$

(which is obtained by decomposition of $[L^n, L^m]_* = 0$), (2.3), and (2.1), we obtain

$$\big((L^m)_+\big)_{t_n} - \big((L^n)_+\big)_{t_m} + [(L^m)_+, (L^n)_+]_* = 0 \tag{2.5}$$

with the help of which the commutativity of the flows (2.2) follows:

$$\begin{aligned} (L_{t_n})_{t_m} - (L_{t_m})_{t_n} = & \big[(L^n)_+, [(L^m)_+, L]_*\big]_* + \big[L, [(L^n)_+, (L^m)_+]_*\big]_* + \\ & + \big[(L^m)_+, [L, (L^n)_+]_*\big]_* \,, \end{aligned} \tag{2.6}$$

which vanishes due to the Jacobi identity. The equations (2.2) and (2.5) are the integrability conditions of the linear system

$$L * \psi = \lambda\, \psi \,, \quad \psi_{t_n} = (L^n)_+ * \psi \,, \quad n = 1, 2, \dots \,, \tag{2.7}$$

where $\lambda_{t_n} = 0$ and ψ lies in a domain on which the algebra $\mathcal{A}$ acts. So far, everything is well-known.

Let us now specify the multiplication in $\mathcal{A}$ as the Moyal product (1.3) with (1.5), and replace (2.1) by the stronger condition

$$\mathcal{A}_- * \mathcal{A}_- \subset \mathcal{A}_- \,, \quad \mathcal{A}_+ * \mathcal{A}_+ \subset \mathcal{A}_+ \,, \tag{2.8}$$

so that $\mathcal{A}_-$ and $\mathcal{A}_+$ are *subalgebras* of $(\mathcal{A}, *)$. We extend the above linear system by appending linear equations of the form

$$\psi_{\theta_{m,n}} = W^{m,n} * \psi \,. \tag{2.9}$$

This gives rise to additional integrability conditions, in particular

$$L_{\theta_{m,n}} = [W^{m,n}, L]_* + \tfrac{1}{2}\big(L_{t_n} * (L^m)_+ - L_{t_m} * (L^n)_+\big) \,, \tag{2.10}$$

where we assumed $\lambda_{\theta_{m,n}} = 0$ and made use of the identity

$$(f * g)_{\theta_{m,n}} = f_{\theta_{m,n}} * g + f * g_{\theta_{m,n}} + \tfrac{1}{2}(f_{t_m} * g_{t_n} - f_{t_n} * g_{t_m}) \tag{2.11}$$

for functions f, g. For later use, we note that (2.10) implies

$$(L^k)_{\theta_{m,n}} = [W^{m,n}, L^k]_* + \tfrac{1}{2}\big((L^k)_{t_n} * (L^m)_+ - (L^k)_{t_m} * (L^n)_+\big) \tag{2.12}$$

(see also [7]). The remaining integrability conditions

$$\begin{aligned} 0 &= (W^{m,n})_{t_k} + [W^{m,n}, (L^k)_+]_* - ((L^k)_+)_{\theta_{m,n}} - \\ &\quad - \tfrac{1}{2}\big(((L^k)_+)_{t_m} * (L^n)_+ - ((L^k)_+)_{t_n} * (L^m)_+ \big)\,, \end{aligned} \tag{2.13}$$

$$\begin{aligned} 0 &= (W^{m,n})_{\theta_{r,s}} - (W^{r,s})_{\theta_{m,n}} + [W^{m,n}, W^{r,s}]_* + \tfrac{1}{2}\big((W^{m,n})_{t_r} * (L^s)_+ - \\ &\quad -(W^{m,n})_{t_s} * (L^r)_+ - (W^{r,s})_{t_m} * (L^n)_+ + (W^{r,s})_{t_n} * (L^m)_+ \big) \end{aligned} \tag{2.14}$$

require the commutativity of all the new flows (2.10) and their commutativity with the flows (2.2). If these equations can be satisfied with a suitable choice of the operators $W^{m,n}$, then equations (2.10) extend the hierarchy (2.2) to a larger hierarchy of mutually commuting flows.

For the moment, let us make the further assumption that

$$(L_{t_n})_+ = 0 = (L_{\theta_{m,n}})_+\,, \tag{2.15}$$

i.e., $L_{t_n}, L_{\theta_{m,n}} \in \mathcal{A}_-$. Then (2.2) implies $([(L^n)_+, L]_*)_+ = 0$ and, using (2.8), (2.10) leads to

$$\begin{aligned} \Big([W^{m,n}, L]_*\Big)_+ &= \tfrac{1}{2}\Big(L_{t_m} * (L^n)_+ - L_{t_n} * (L^m)_+\Big)_+ = \\ &= \tfrac{1}{2}\Big([(L^m)_+, L]_* * L^n - [(L^n)_+, L]_* * L^m\Big)_+ = \\ &= \tfrac{1}{2}\Big([(L^m)_+ * L^n - (L^n)_+ * L^m, L]_*\Big)_+ = \\ &= \tfrac{1}{2}\Big([(L^n)_- * L^m - (L^m)_- * L^n, L]_*\Big)_+\,, \end{aligned} \tag{2.16}$$

so that

$$\Big(\Big[W^{m,n} - \tfrac{1}{2}\big((L^n)_- * L^m - (L^m)_- * L^n\big), L\Big]_*\Big)_+ = 0\,. \tag{2.17}$$

Assuming moreover

$$[L, \mathcal{A}_-]_* \subset \mathcal{A}_- \tag{2.18}$$

(in accordance with (2.2) and (2.15)), we may take $W^{m,n} \in \mathcal{A}_+$. The last equation is then satisfied if

$$\begin{aligned} W^{m,n} &= \tfrac{1}{2}\big((L^n)_- * L^m - (L^m)_- * L^n\big)_+ = \\ &= \tfrac{1}{2}\big((L^n)_- * (L^m)_+ - (L^m)_- * (L^n)_+\big)_+\,. \end{aligned} \tag{2.19}$$

It turns out that, with this choice, the flows (2.2) and (2.10) indeed commute, so that the integrability conditions of the extended linear system are satisfied.

Theorem. Let $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ with subalgebras $\mathcal{A}_\pm$, and let $W^{m,n}$ be given by (2.19). The equations (2.13) and (2.14) are then consequences of (2.2) and (2.10).

Proof: Using (2.12), (2.3), (2.19) and (2.8), a straightforward calculation leads to

$$\begin{aligned}&((L^k)_+)_{\theta_{m,n}} - [W^{m,n},(L^k)_+]_* + \tfrac{1}{2}\big(((L^k)_+)_{t_m} * (L^n)_+ - ((L^k)_+)_{t_n} * (L^m)_+ \big) = \\ &= \tfrac{1}{2}\big(([(L^n)_-,(L^k)_-]_* + [(L^n)_+,(L^k)_-]_*)_- * (L^m)_+ - \\ &\quad -([(L^m)_-,(L^k)_-]_* + [(L^m)_+,(L^k)_-]_*)_- * (L^n)_+ + \\ &\quad +(L^n)_- * [(L^m)_+,(L^k)_-]_* - (L^m)_- * [(L^n)_+,(L^k)_-]_* \big)_+ .\end{aligned}$$

By use of the identity (2.4) this equals

$$\begin{aligned}(W^{m,n})_{t_k} = &\big([(L^m)_-,(L^k)_+]_* * (L^n)_+ - [(L^n)_-,(L^k)_+]_* * (L^m)_+ + \\ &+(L^m)_- * [L^n,(L^k)_+]_* - (L^n)_- * [L^m,(L^k)_+]_* \big)_+ ,\end{aligned}$$

so that (2.13) holds. A similar, but tedious calculation, shows that (2.14) is also satisfied. ■

Note that our intermediate assumptions (2.15) and (2.18) were not needed in obtaining this result. Of course, only with a suitable choice of $\mathcal{A}$ and L, the construction in this section will lead to meaningful results. One has to ensure that (2.2) and (2.10) are sufficiently free of constraints. Examples will be presented in the following sections.

2.1 $N-$reductions

Using (2.3), (2.12), and (2.8), we obtain

$$((L^N)_-)_{t_n} = ((L^N)_{t_n})_- = ([(L^n)_+, L^N]_*)_- , \tag{2.20}$$

$$\begin{aligned}((L^N)_-)_{\theta_{m,n}} &= ((L^N)_{\theta_{m,n}})_- = \\ &= \tfrac{1}{2}\big([(L^n)_+,(L^N)_-]_* * (L^m)_+ - [(L^m)_+,(L^N)_-]_* * (L^n)_+\big)_- + \\ &\quad +([W^{m,n},(L^N)_-]_*)_- ,\end{aligned} \tag{2.21}$$

which shows that the extended hierarchy preserves the *N–reduction* constraint (see [2], for example)

$$(L^N)_- = 0 , \quad N \in \mathbb{N} . \tag{2.22}$$

This leads to

$$\begin{aligned}&L_{t_{kN}} = 0 , \qquad L_{\theta_{kN,lN}} = 0 , \qquad \forall k,l \in \mathbb{N} , \\ &L_{\theta_{kN,lN+r}} = \tfrac{1}{2} L_{t_{(k+l)N+r}} \quad \forall k \in \mathbb{N} ,\ \ l \in \mathbb{N}\cup\{0\} ,\ \ r = 1,2,\ldots,N-1 .\end{aligned} \tag{2.23}$$

In particular, after reduction, on the variables $t_{(k+l)N+r}$ and $\theta_{kN,lN+r}$ the fields only depend through the combination $t_{(k+l)N+r} + \frac{1}{2}\theta_{kN,lN+r}$.

3 Extension of Lax hierarchies in Sato theory

Let $\mathcal{A}$ be the algebra of (formal) pseudo–differential operators, the elements of which are formal series in integer powers of ∂ (the derivation with respect to

$x = t_1$). The coefficients of powers of ∂ are taken to be matrices of functions of variables $t_1, t_2, \ldots$, where $t_1 = x$. The action of powers of the formal inverse ∂^{-1} on functions is given by

$$\partial^{-m} f = \sum_{j=0}^{\infty} (-1)^j \binom{m+j-1}{j} \frac{\partial^j f}{\partial x^j} \, \partial^{-m-j}, \quad m > 0 \tag{3.1}$$

(see [2], for example). For $k \in \mathbb{Z}$, the algebra $\mathcal{A}$ splits according to

$$\mathcal{A} = \mathcal{A}_{\geq k} \oplus \mathcal{A}_{<k} \,, \tag{3.2}$$

where elements of $\mathcal{A}_{\geq k}$ only contain powers $\geq k$ of ∂, whereas those of $\mathcal{A}_{<k}$ only contain powers $< k$. But only for $k = 0$ and $k = 1$ this decomposition satisfies (2.1), i.e. $\mathcal{A}_{\geq k}$ and $\mathcal{A}_{<k}$ are Lie subalgebras in these cases.[2]) Choosing

$$L = \begin{cases} \partial + u_2 \, \partial^{-1} + u_3 \, \partial^{-2} + \ldots & \text{if } \ k = 0 \,, \\ \partial + u_1 + u_2 \, \partial^{-1} + u_3 \, \partial^{-2} + \ldots & \text{if } \ k = 1 \end{cases} \tag{3.3}$$

the equations

$$L_{t_n} = [(L^n)_{\geq k}, L]_* \,, \quad n = 1, 2, \ldots \,, \tag{3.4}$$

where $(\ )_{\geq k}$ projects to $\mathcal{A}_{\geq k}$, determine the *ncKP* ($k = 0$) and *ncmKP* ($k = 1$) hierarchies (see [1], for example, for the commutative case, and [5, 9, 10, 11, 12]). Since for $k = 0, 1$, $\mathcal{A}_{\geq k}$ and $\mathcal{A}_{<k}$ are moreover subalgebras of $(\mathcal{A}, *)$, the theorem in section 2 applies. We thus obtain extensions of the corresponding hierarchies, which we call *xncKP* (see also [7, 8]) and *xncmKP*, respectively. In the following, k is either 0 or 1. Using the above form of L, the identity $[(L^n)_{\geq k}, L]_* = -[(L^n)_{<k}, L]_*$ shows that the right hand side of (3.4) only contains powers of ∂ up to the order -1 if $k = 0$ and up to order 0 if $k = 1$, which is consistent with the left hand side.

Let us introduce the (formal) pseudo–differential operator

$$X = w_0 + \sum_{m=1}^{\infty} w_m \, \partial^{-m} \,, \tag{3.5}$$

with (matrices of) functions w_m depending on the variables t_n, $n \in \mathbb{N}$. Setting $w_0 = 1$ if $k = 0$ and assuming w_0 to be invertible if $k = 1$, the ncKP and ncmKP hierarchy can be introduced via[3])

$$X_{t_n} = -(L^n)_{<k} * X \,, \quad n = 1, 2, \ldots \,, \quad k \in \{0, 1\} \,, \tag{3.6}$$

[2]) In the case of *commutative* coefficients, one obtains also for $k = 2$ a decomposition into Lie subalgebras (see [1], for example). With *non*commutative coefficients this is no longer so, since in general the commutator of two vector fields is no longer a vector field: $[u \, \partial, v \, \partial]_* = (u * v_x - v * u_x) \, \partial + [u, v]_* \, \partial^2$. As a consequence, there is no *non*commutative version of the Harry–Dym hierarchy (where $L = u_0 \, \partial + u_1 + u_2 \, \partial^{-1} + u_3 \, \partial^{-2} + \ldots$), at least not in the framework considered here.

[3]) For $k = 0$, this equation apparently appeared first in [13]. See also [14, 15, 16] and, in particular, [17], where the $k = 0, 1$ *matrix* hierarchies are discussed.

where[4])

$$L = X * \partial\, X^{-1}\,. \tag{3.7}$$

The (formal) Baker–Akhiezer function

$$\psi = X * \mathrm{e}^{\sum_{n=1}^{\infty} t_n \lambda^n} \tag{3.8}$$

(with a parameter λ) then satisfies

$$L * \psi = \lambda\psi\,, \quad \psi_{t_n} = (L^n)_{\geq k} * \psi\,, \quad n = 1, 2, \ldots\,, \quad k \in \{0, 1\}\,. \tag{3.9}$$

Conversely, (3.6) results as compatibility condition of this linear system.

Equations (3.6) and (3.7) imply (3.4). According to the discussion in the previous section, the linear system is consistently extended by (2.9) with

$$W^{m,n} = \tfrac{1}{2}\big((L^n)_{<k} * L^m - (L^m)_{<k} * L^n\big)_{\geq k}\,. \tag{3.10}$$

In terms of X, the deformation equations which extend the nc(m)KP hierarchy are given by

$$X_{\theta_{m,n}} = -\tfrac{1}{2}\big((L^n)_{<k} * L^m - (L^m)_{<k} * L^n\big)_{<k} * X \tag{3.11}$$

(see [8] for details in the xncKP case $k = 0$).

3.1 Birkhoff factorization for the xncKP and xncmKP hierarchy

The nc(m)KP equation (3.6) can be rewritten as[5])

$$X_{t_n} = -L^n * X + (L^n)_{\geq k} * X = (L^n)_{\geq k} * X - X\,\partial^n\,. \tag{3.12}$$

Introducing

$$\hat{\xi} = \sum_{n=1}^{\infty} t_n\, \partial^n\,, \tag{3.13}$$

this can be expressed as follows,

$$\big(X * \mathrm{e}^{\hat{\xi}}\big)_{t_n} = L^n{}_{\geq k} * \big(X * \mathrm{e}^{\hat{\xi}}\big)\,. \tag{3.14}$$

In the same way, (3.11) leads to

$$\begin{aligned} X_{\theta_{m,n}} &= W^{m,n} * X - \tfrac{1}{2}\big((L^n)_{<k} * X\,\partial^m - (L^m)_{<k} * X\,\partial^n\big) = \\ &= W^{m,n} * X + \tfrac{1}{2}(X_{t_n} * \partial^m - X_{t_m} * \partial^n) \end{aligned} \tag{3.15}$$

and thus

$$\big(X * \mathrm{e}^{\hat{\xi}}\big)_{\theta_{m,n}} = W^{m,n} * \big(X * \mathrm{e}^{\hat{\xi}}\big)\,. \tag{3.16}$$

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[4]) Equation (3.7) actually follows from (3.6) with $n = 1$: $-L_{<k} * X = X_x = [\partial, X]_* = L_{\geq k} * X - X * \partial$. Hence $L * X = X * \partial$.

[5]) See [15] for the case $k = 0$. In [18], it has been called *Sato equation*.

Theorem (factorization).[6] The xnc(m)KP equations (3.6) and (3.11) are equivalent to[7]

$$X * \mathrm{e}^{\hat{\xi}} * X_0^{-1} \in G_+ \,, \tag{3.17}$$

where X_0 is the initial value of X at $t := \{t_n\}_{n=1}^{\infty} = 0$, $\theta := \{\theta_{m,n}\}_{m,n=1}^{\infty} = 0$, and G_+ is the formal group[8] generated by the subalgebra $\mathcal{A}_{\geq k}$ (where $k \in \{0,1\}$).

Proof: Let us assume that (3.6) and (3.11) hold. We have already shown that these equations lead to (3.14) and (3.16). With the help of (3.6) and (3.11), the latter equations imply

$$\frac{\partial}{\partial t_{p_1}} \cdots \frac{\partial}{\partial t_{p_l}} \frac{\partial}{\partial \theta_{m_1,n_1}} \cdots \frac{\partial}{\partial \theta_{m_r,n_s}} \left(X * \mathrm{e}^{\hat{\xi}}\right) = B_{p_1,\ldots,p_l;m_1,n_1,\ldots,m_r,n_s} * \left(X * \mathrm{e}^{\hat{\xi}}\right) ,$$

where $B_{p_1,\ldots,p_l;m_1,n_1,\ldots,m_r,n_s} \in \mathcal{A}_{\geq k}$. Evaluated at $t = 0$, $\theta = 0$, these expressions determine (formal) Taylor coefficients of $X * \mathrm{e}^{\hat{\xi}}$. Hence $X * \mathrm{e}^{\hat{\xi}} = Y * X_0$ with

$$Y = 1 + \sum_{n=1}^{\infty} t_n \, B_n\Big|_{t=0,\theta=0} + \sum_{m<n} \theta_{m,n} \, B_{m,n}\Big|_{t=0,\theta=0} + \ldots \,,$$

where all higher order terms are also in $\mathcal{A}_{\geq k}$. Hence Y is an invertible element of $\mathcal{A}_{\geq k}$. Furthermore, $Y \in G_+$, since the formal group G_+ is defined as the set of formal power series Y (in the variables $t_p, \theta_{m,n}$), such that $Y - 1 \in \mathcal{A}_{\geq k}$ and $Y - 1$ vanishes at $t = 0, \theta = 0$.[9] Thus (3.17) holds.

Conversely, let us assume that X satisfies (3.17) and denote its left hand side by Y, so that

$$X * \mathrm{e}^{\hat{\xi}} = Y * X_0 \,.$$

Differentiation with respect to t_m leads to

$$(X_{t_m} + X \, \partial^m) * \mathrm{e}^{\hat{\xi}} = (X * \mathrm{e}^{\hat{\xi}})_{t_m} = Y_{t_m} * X_0 = Y_{t_m} * Y^{-1} * X * \mathrm{e}^{\hat{\xi}}$$

and thus

$$X_{t_m} * X^{-1} + X * \partial^m * X^{-1} = Y_{t_m} * Y^{-1} \,.$$

The right hand side lies in $\mathcal{A}_{\geq k}$, since $Y \in G_+$ and $\mathcal{A}_{\geq k} * G_+ \subset \mathcal{A}_{\geq k}$. Since $X_{t_m} * X^{-1} \in \mathcal{A}_{<k}$ (see (3.5)), by taking the $\mathcal{A}_{<k}$–part of the last equation we find

$$X_{t_m} = -(X * \partial^m * X^{-1})_{<k} * X \,,$$

which is (3.6). In the same way, we obtain

$$\begin{aligned} \left(X_{\theta_{m,n}} + \tfrac{1}{2}(X_{t_m} \, \partial^n - X_{t_n} \, \partial^m)\right) * \mathrm{e}^{\hat{\xi}} &= \left(X * \mathrm{e}^{\hat{\xi}}\right)_{\theta_{m,n}} = \\ &= Y_{\theta_{m,n}} * X_0 = \\ &= Y_{\theta_{m,n}} * Y^{-1} * X * \mathrm{e}^{\hat{\xi}} \end{aligned}$$

[6]) See also [16, 19, 12].

[7]) Written in the form $\exp(\hat{\xi}) * X_0^{-1} = X^{-1} * Y$, this is the *Birkhoff factorization* (generalized Riemann–Hilbert problem) of $\exp(\hat{\xi}) * X_0^{-1}$. A group element is written in a unique way as the product of an element of G_- (here the formal group generated by $\mathcal{A}_{<k}$) and an element of G_+.

[8]) See [16], in particular.

[9]) Correspondingly, $Z \in G_-$ is defined in the same way, but with $Z - 1 \in \mathcal{A}_{<k}$. See also [16].

and therefore

$$\big(X_{\theta_{m,n}} + \tfrac{1}{2}(X_{t_m}\,\partial^n - X_{t_n}\,\partial^m\big) * X^{-1} = Y_{\theta_{m,n}} * Y^{-1}\,.$$

Taking the $\mathcal{A}_{<k}$–part of this equation yields

$$X_{\theta_{m,n}} = -\tfrac{1}{2}(X_{t_m} * \partial^n * X^{-1} - X_{t_n} * \partial^m * X^{-1})_{<k} * X\,,$$

which, by use of (3.6), becomes (3.11). Thus, (3.17) implies (3.6) and (3.11). ■

This result also shows that the deformation equations which extend the Moyal–deformed hierarchy are in fact a natural property of the latter.

3.2 xncKP more explicitly

In this subsection, we take a closer look at the case $k = 0$.[10]) In terms of

$$w_1 = -\phi \tag{3.18}$$

the following formulae for the xncKP hierarchy equations were obtained in [8]:

$$\phi_{t_m\,t_n} = \sigma^{(n)}_{m+1} + (\sigma^{(n)}_{m+1})^\omega + \sum_{i=1}^{m-1}\big(\sigma^{(n)}_{m-i} * \phi_{t_i} + \phi_{t_i} * (\sigma^{(n)}_{m-i})^\omega\big)\,, \tag{3.19}$$

$$\begin{aligned}\phi_{\theta_{m,n}} = -\tfrac{1}{2}\Big(&\phi_{t_{m+n}} + \sigma^{(n)}_{m+1} - (\sigma^{(n)}_{m+1})^\omega + \\ &+ \sum_{i=1}^{m-1}\big(\sigma^{(n)}_{m-i} * \phi_{t_i} - \phi_{t_i} * (\sigma^{(n)}_{m-i})^\omega\big)\Big)\,.\end{aligned} \tag{3.20}$$

The coefficients $\sigma^{(n)}_m$ are determined iteratively by

$$\sigma^{(n+1)}_m = \sigma^{(1)}_{m,t_n} + \sigma^{(n)}_{m+1} + \sigma^{(1)}_{n+m} - \sum_{j=1}^{n-1}\sigma^{(1)}_j * \sigma^{(n-j)}_m + \sum_{j=1}^{m-1}\sigma^{(1)}_j * \sigma^{(n)}_{m-j} \tag{3.21}$$

and

$$\sigma^{(1)}_n = p_n(-\tilde{\partial})\phi\,, \tag{3.22}$$

with $\tilde{\partial} = (\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \frac{1}{3}\partial_{t_3}, \ldots)$ and the Schur polynomials

$$p_n(t_1,t_2,t_3,\ldots) = \sum_{\substack{m_1+2\,m_2+\ldots+n\,m_n=n\\ m_i\geq 0}} \frac{1}{m_1!\cdots m_n!}\,t_1^{m_1}\cdots t_n^{m_n}\,. \tag{3.23}$$

[10]) Using $L_{t_2} = [(L^2)_{\geq k}, L]_*$, in the $k=0$ case all the coefficients u_j of L with $j>2$ can be expressed (via an x–integration) in terms of u_2 $(=\phi_x)$, so that the whole hierarchy becomes a set of equations for a single dependent variable. This does not work in the *non*commutative $k=1$ case. For example, one obtains $u_{1,t_2} = u_{1,xx} + 2\,u_1 * u_{1,x} + 2\,u_{2,x} + 2\,[u_1,u_2]_*$, which cannot be solved for u_2 in terms of u_1 and its derivatives by an x–integration, since the commutator term also involves u_2. See also [9].

Furthermore, $^\omega$ is an involution determined by

$$(f_{t_n})^\omega = -(f^\omega)_{t_n}\,, \quad (f_{\theta_{m,n}})^\omega = -(f^\omega)_{\theta_{m,n}}\,, \quad (f*h)^\omega = -h^\omega * f^\omega\,, \quad \phi^\omega = \phi\,. \tag{3.24}$$

From these formulae we obtain in particular[11])

$$(\phi_{t_3})_x = \tfrac{1}{4}\Big(3\,\phi_{yy} + \phi_{xxxx} - 6\,[\phi_x, \phi_y]_* + 6\,({\phi_x}^2)_x\Big), \tag{3.25}$$

$$(\phi_{t_4})_x = \tfrac{1}{3}\Big(2\,\phi_{yt_3} + \phi_{xxxy} - [\phi_x, 4\,\phi_{t_3} - \phi_{xxx}]_* + 3\,(\{\phi_x, \phi_y\}_*)_x\Big), \tag{3.26}$$

$$\begin{aligned}(\phi_{t_5})_x = \tfrac{1}{216}\Big(&90\,\phi_{yt_4} + 40\,\phi_{t_3t_3} + 40\,\phi_{t_3xxx} + 45\,\phi_{xxyy} + \phi_{xxxxxx} + \\ &+270\,[\phi_{t_4}, \phi_x]_* + 60\,[\phi_{t_3}, \phi_y + 3\,\phi_{xx}]_* + 120\,\{\phi_{xt_3}, \phi_x\}_* + \\ &+45\,\{\phi_{yy}, \phi_x\}_* + 180\,\{\phi_{xy}, \phi_y\}_* + 60\,[\phi_y, \phi_{xxx}]_* + \\ &+90\,[\phi_x, \phi_{xxy}]_* + 15\,\{\phi_{xxxx}, \phi_x\}_*\Big)\end{aligned} \tag{3.27}$$

and

$$\phi_{\theta_{1,2}} = \tfrac{1}{6}(\phi_{t_3} - \phi_{xxx}) - {\phi_x}^2\,, \tag{3.28}$$

$$\phi_{\theta_{1,3}} = \tfrac{1}{4}\Big(\phi_{t_4} - \phi_{xxy} - 3\,\{\phi_x, \phi_y\}_* - [\phi_x, \phi_{xx}]_*\Big), \tag{3.29}$$

$$\begin{aligned}\phi_{\theta_{1,4}} = &\tfrac{3}{10}\phi_{t_5} - \tfrac{1}{6}\phi_{xxt_3} - \tfrac{1}{8}\phi_{xyy} - \tfrac{1}{120}\phi_{xxxxx} - \tfrac{2}{3}\{\phi_x, \phi_{t_3}\}_* - \\ &-\tfrac{1}{12}\{\phi_x, \phi_{xxx}\}_* - \tfrac{1}{4}[\phi_x, \phi_{xy}]_* + \tfrac{1}{4}[\phi_{xx}, \phi_y]_* - \tfrac{1}{2}{\phi_y}^2\,,\end{aligned} \tag{3.30}$$

$$\begin{aligned}\phi_{\theta_{2,3}} = &\tfrac{1}{10}\phi_{t_5} - \tfrac{1}{8}\phi_{xyy} + \tfrac{1}{40}\phi_{xxxxx} - \tfrac{3}{4}{\phi_y}^2 - \tfrac{1}{4}[\phi_x, \phi_{xy}]_* + \\ &+\tfrac{1}{4}\{\phi_x, \phi_{xxx}\}_* + \tfrac{1}{4}{\phi_{xx}}^2 + {\phi_x}^3\end{aligned} \tag{3.31}$$

(see also [8]). Here $\{\ ,\ \}_*$ is the anti–commutator in the $*$–product algebra, and we wrote y instead of t_2. Common N–soliton solutions of these equations were obtained in [8] (see also [12, 20, 21]).

3.3 $N-$reductions of the xnc(m)KP hierarchy

Let us consider the N–reduction constraint

$$(L^N)_{<k} = 0\,, \quad N \in \mathbb{N}\,, \quad k \in \{0, 1\} \tag{3.32}$$

(cf section 2.1). In the case $k = 0$ it reduces the (non-deformed) KP hierarchy to the Nth Gelfand–Dickey (GD) hierarchy [2]. Writing

$$(L^N)_{\geq 0} = \partial^N + v_{N-2}\,\partial^{N-2} + v_{N-3}\,\partial^{N-3} + \ldots + v_0\,, \tag{3.33}$$

with (matrices of) functions v_i, the w_j can be expressed as differential polynomials of the v_i (see [2], for example). In the case $k = 1$, we have

$$(L^N)_{\geq 1} = \partial^N + v_{N-1}\,\partial^{N-1} + v_{N-2}\,\partial^{N-2} + \ldots + v_1\,\partial \tag{3.34}$$

[11]) Here ${\phi_x}^2$, for example, stands for $\phi_x * \phi_x$. In our previous papers [7, 8] we wrote ${\phi_x}^{*2}$ instead.

instead. The xnc(m)KP equations are then reduced to

$$(L^N)_{t_n} = [(L^n)_{\geq k}, L^N]_* \quad \forall\, n \in \mathbb{N}\,, \quad n/N \notin \mathbb{N}\,, \tag{3.35}$$

$$(L^N)_{\theta_{m,n}} = \tfrac{1}{2}\big((L^N)_{t_n} * (L^m)_{\geq k} - (L^N)_{t_m} * (L^n)_{\geq k}\big) + [W^{m,n}, L^N]_*\,, \quad \forall\, m,n \in \mathbb{N}\,, \quad m/N\,,\ n/N \notin \mathbb{N}\,. \tag{3.36}$$

For $k=0$, this yields a recipe to obtain the equations of the reduced xncKP hierarchy easily from those of the xncKP hierarchy presented in the previous subsection in terms of the potential ϕ. It amounts to allowing ϕ only to depend on t_{jN+r} and $\theta_{jN+r,lN+s}$, where $j,l = 0,1,2,\ldots$ and $r,s = 1,2,\ldots,N-1$. Furthermore, derivatives of ϕ with respect to t_{mN}, $m \in \mathbb{N}$, have to be dropped, and a derivative with respect to $\theta_{jN,lN+r}$, where $j \in \mathbb{N}$ and $l \in \mathbb{N}\cup\{0\}$, has to be replaced by $1/2$ times the derivative with respect to $t_{(j+l)N+r}$. In particular the last feature provides us with a new way to obtain the flows corresponding to the t_n with $n > N$ of the Nth GD hierarchy by extending the Moyal–deformed KP hierarchy, applying the reduction conditions, and then returning to vanishing deformation:

$$\begin{array}{ccc} \text{ncKP} & \longrightarrow & \text{deformation equations} \\ \uparrow & & \downarrow \\ \text{KP} & \longrightarrow & N\text{th GD hierarchy} \end{array}$$

Of course, the above method also leads to *extensions* of the Moyal–deformed GD hierarchies. The cases $N=2$ (xncKdV hierarchy) and $N=3$ (xncBoussinesq hierarchy) were treated in [8]. For $N=4$, writing

$$L^4 = \partial^4 + u\,\partial^2 + v\,\partial + w\,, \tag{3.37}$$

with (matrices of) functions u, v, w, the reduction condition implies $u = -4w_{1,x}$ and (3.35) with $n=2$ leads to

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_y = \begin{pmatrix} 2\,(v_x - u_{xx}) \\ 2\,w_x + v_{xx} - 2\,u_{xxx} - u\,u_x + \frac{1}{2}[u,v]_* \\ w_{xx} - \frac{1}{2}(v\,u_x + u\,u_{xx} + u_{xxxx}) + \frac{1}{2}[u,w]_* \end{pmatrix} \tag{3.38}$$

(see [1], for example, for the commutative case). Elimination of v, w, and introduction of the potential ϕ, using $u = 4\phi_x$, leads to the integro–differential equation

$$\begin{aligned} &\big(D^{-1}\phi_{yy} + \phi_{xxx} + 4{\phi_x}^2 - 2D^{-1}[\phi_x,\phi_y]_*\big)_y + 2\{\phi_{xx},\phi_y\}_* - \\ &\quad -2[\phi_x, D^{-1}\phi_{yy}]_* + 4\big[\phi_x, D^{-1}[\phi_x,\phi_y]_*\big]_* = 0\,, \end{aligned} \tag{3.39}$$

where D^{-1} denotes integration with respect to x. Alternatively, we obtain the last equation more directly from (3.25) and (3.26) (where $\phi_{t_4} = 0$ as a consequence of the reduction conditions) by elimination of ϕ_{t_3}, which involves an integration. Similarly, the deformation equations of the reduced hierarchy are either obtained from (3.36), or more directly from equations like (3.28), (3.29) etc:

$$\phi_{\theta_{1,2}} = \tfrac{1}{8}\big(D^{-1}\phi_{yy} - \phi_{xxx} - 6{\phi_x}^2 - 2D^{-1}[\phi_x,\phi_y]_*\big)\,, \tag{3.40}$$

$$\phi_{\theta_{1,3}} = -\tfrac{1}{4}\big(\phi_{xxy} + 3\{\phi_x,\phi_y\}_* + [\phi_x,\phi_{xx}]_*\big)\,. \tag{3.41}$$

See also [9, 10] for some other reductions of the ncKP and ncmKP hierarchies.

4 Extended ncToda lattice hierarchy

Let S denote the shift operator $(Sf)_k = f_{k+1}$ acting on functions which depend on a discrete variable $k \in \mathbb{Z}$. Let $\mathcal{A}$ be the algebra of formal series in S, the coefficients of which are matrices of functions of variables $t_1, t_2, \ldots$. Again, multiplication should obey the Moyal product rule (1.3) with (1.5). We have the decomposition

$$\mathcal{A} = \mathcal{A}_{\geq 0} \oplus \mathcal{A}_{<0} \,, \tag{4.1}$$

where $\mathcal{A}_{\geq 0}$ and $\mathcal{A}_{<0}$ are the subalgebras of series with only non-negative, respectively only negative powers of S.

With the choice of Lax operator

$$L = aS^{-1} + b + S \,, \tag{4.2}$$

the *ncToda lattice hierarchy* (see [1], for example, for the commutative case, and [12]) is the system of equations[12])

$$L_{t_n} = [(L^n)_{\geq 0}, L]_* \,, \quad n = 1, 2, 3, \ldots \,, \tag{4.3}$$

where $(\;)_{\geq 0}$ (respectively $(\;)_{<0}$) projects to $\mathcal{A}_{\geq 0}$ (respectively $\mathcal{A}_{<0}$). The coefficient of S^l, $l \in \mathbb{Z}$, is taken by $(\;)_{(l)}$. Since the decomposition satisfies (2.8), it follows that the flows given by (4.3) commute. Furthermore, the theorem in section 2 applies, so that an extension of the hierarchy in the sense of section 2 exists: *xncToda*.

One obtains the following system of equations:

$$\begin{aligned}
a_{t_1} &= b * a - a * b_{-1} \,, \\
b_{t_1} &= a_{+1} - a \,, \\
a_{t_2} &= (a_{+1} + b^2) * a - a * (a_{-1} + {b_{-1}}^2) \,, \\
b_{t_2} &= a_{+1} * b + b_{+1} * a_{+1} - a * b_{-1} - b * a \,, \\
a_{t_3} &= (a_{+1} * b + b * a + b * a_{+1} + b_{+1} * a_{+1} + b^3) * a - \\
&\quad - a * (a_{-1} * b_{-2} + a_{-1} * b_{-1} + a * b_{-1} + b_{-1} * a_{-1} + {b_{-1}}^3) \,, \\
b_{t_3} &= a_{+2} * a_{+1} + {a_{+1}}^2 + a_{+1} * b^2 + b_{+1} * a_{+1} * b + {b_{+1}}^2 * a_{+1} - \\
&\quad - a * a_{-1} - a^2 - a * {b_{-1}}^2 - b * a * b_{-1} - b^2 * a \,, \\
&\;\vdots
\end{aligned} \tag{4.4}$$

where a_{+1}, a_{-1} abbreviate the actions of S, respectively S^{-1}, on a.

In order to compute the θ-equations, which extend the ncToda hierarchy, we have to evaluate

$$W^{m,n} = \tfrac{1}{2}\big((L^n)_{<0} * L^m - (L^m)_{<0} * L^n\big)_{\geq 0} \,. \tag{4.5}$$

[12]) In the identity $[(L^n)_{\geq 0}, L]_* = -[(L^n)_{<0}, L]_*$, the left hand side only contains the powers $-1, \ldots, n+1$ of S, whereas on the right side only the powers $-n-1, \ldots, 0$ appear. As a consequence, only the coefficients of S^{-1} and S^0 survive. This is consistent with the left hand side of (4.3) due to the form (4.2) of the Lax operator.

In particular, we find

$$
\begin{aligned}
W^{1,2} &= \tfrac{1}{2}[b,a]_* - \tfrac{1}{2}aS\,, && (4.6)\\
W^{1,3} &= \tfrac{1}{2}([a_{+1}+b^2,a]_* + [b,a*b_{-1}]_* - \tfrac{1}{2}a*(b_{-1}+b+b_{+1})S - \tfrac{1}{2}aS^2\,, && (4.7)\\
W^{2,3} &= \tfrac{1}{2}(a*[a,b_{-1}]_* + [a^2,b]_* + [a_{+1}+b^2,a*b_{-1}]_* - \\
&\quad -b*[a,b]_**b + a_{+1}*a*b - b*a*a_{+1}) + \\
&\quad +\tfrac{1}{2}\big(a^2 + a_{+1}*a - (a*b_{-1}+b*a)*(b+b_{+1}) - b^2*a\big)S - \\
&\quad -\tfrac{1}{2}\big(a*b_{-1}+b*a\big)S^2\,. && (4.8)
\end{aligned}
$$

Now (2.10) in particular leads to

$$
\begin{aligned}
a_{\theta_{1,2}} &= \tfrac{1}{2}\big(a*b_{-1}*a - a*b*a + a_{+1}*a*b_{-1} - b*a*a_{-1} - \\
&\quad -b*a*{b_{-1}}^2 + b^2*a*b_{-1}\big)\,, && (4.9)\\
b_{\theta_{1,2}} &= \tfrac{1}{2}\big(a^2 - {a_{+1}}^2 - b*a*b_{-1} + b_{+1}*a_{+1}*b\big)\,. && (4.10)
\end{aligned}
$$

5 Remarks

1. Integrable systems where the dependent variables take values in a Moyal algebra (which arises in particular as a large N–limit of models where the dependent variables have values in a Lie algebra of $(N \times N)$–matrices) have been studied extensively some time ago, see [19, 22], for example. In these models, the Moyal product does *not* involve the 'space-time' coordinates, i.e., the evolution parameters and space coordinates. In contrast, more recent work concentrated on Moyal–deformation of the space–time coordinates. In the case of a hierarchy with fields depending on evolution parameters t_1, t_2, ... and corresponding deformation parameters $\theta_{m,n}$, one may consider sub–hierarchies where indeed the Moyal–product only involves a subset of the t_n which is disjoint from the subset of 'evolution times' of the respective sub–hierarchy. In this sense we are indeed led to Moyal models of the older kind.

2. The formalism of section 2 can still be generalized, as becomes apparent from the work in [7]. Let $\mathcal{A}$ be the algebra of formal (integer power) series in a variable λ with coefficients multiplied according to the Moyal product rule (1.3) with (1.5). We have the decomposition

$$\mathcal{A} = \mathcal{A}_{\geq 0} \oplus \mathcal{A}_{<0} \tag{5.1}$$

into the subalgebras of non-negative, respectively negative powers of λ. Instead of (2.2) we set

$$L_{t_n} = [(\lambda^n\, L)_{\geq 0}, L]_*\,. \tag{5.2}$$

Choosing

$$W^{m,n} = \tfrac{1}{2}\big((\lambda^n\, L)_{<0} * \lambda^m\, L - (\lambda^m\, L)_{<0} * \lambda^n\, L\big)_{\geq 0} \tag{5.3}$$

it has been shown [7] that

$$L_{\theta_{m,n}} = \big[W^{m,n}, L\big]_* + \tfrac{1}{2}\big(L_{t_n} * (\lambda^m\, L)_{\geq 0} - L_{t_m} * (\lambda^n\, L)_{\geq 0}\big) \tag{5.4}$$

extends the hierarchy (5.2) to a larger hierarchy. We thus recover precisely the corresponding formulae of section 2 with the formal replacement $\lambda^n L \mapsto L^n$. A concrete example of the above structure is given by the extended Moyal–deformed AKNS hierarchy [7].

3. A suitable framework for further generalizations can be formulated as follows. Let $\mathcal{B}$ be an abelian algebra of linear operators acting on an associative algebra $(\mathcal{A}, *)$ such that

$$\Theta \circ \mathbf{m}_* = \mathbf{m}_* \circ \Delta(\Theta) \,, \quad \forall\, \Theta \in \mathcal{B} \,, \tag{5.5}$$

with the product map $\mathbf{m}_* : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, and a coassociative coproduct of the form

$$\Delta(\Theta) = 1 \otimes \Theta + \Theta \otimes 1 + \Delta'(\Theta) \,, \quad \Delta'(\Theta) = \sum \Theta_{(1)} \otimes \Theta_{(2)} \,, \tag{5.6}$$

with $\Theta_{(1)}, \Theta_{(2)} \in \mathcal{B}$ (using the Sweedler notation, see [23], for example). Furthermore, let $\mathcal{L} : \mathcal{B} \to \mathcal{A}$ be a linear map. The integrability conditions of the linear system

$$L * \psi = \lambda \psi \,, \quad \Theta \psi = \mathcal{L}(\Theta) * \psi \,, \quad \forall \Theta \in \mathcal{B} \,, \tag{5.7}$$

with[13]) $L, \psi \in \mathcal{A}$ are then

$$\Theta\, L = [\mathcal{L}(\Theta), L]_* - \sum (\Theta_{(1)} L) * \mathcal{L}(\Theta_{(2)}) \tag{5.8}$$

and

$$\begin{aligned} \Theta\mathcal{L}(\Theta') - \Theta\mathcal{L}(\Theta') = & \big[\mathcal{L}(\Theta), \mathcal{L}(\Theta')\big]_* - \sum \big(\Theta_{(1)}\mathcal{L}(\Theta')\big) * \mathcal{L}(\Theta_{(2)}) + \\ & + \sum \big(\Theta'_{(1)}\mathcal{L}(\Theta)\big) * \mathcal{L}(\Theta'_{(2)}) \end{aligned} \tag{5.9}$$

for all $\Theta, \Theta' \in \mathcal{B}$. The last equation implies that the Θ–flow given by (5.8) commutes with the corresponding flow associated with Θ'.

In section 2, where $*$ is taken to be the Moyal product, the algebra $\mathcal{B}$ consists of the partial derivative operators $\partial_{t_p}, \partial_{\theta_{m,n}}$, and the map $\mathcal{L}$ is determined by $\mathcal{L}(\partial_{t_p}) = (L^p)_+$ and $\mathcal{L}(\partial_{\theta_{m,n}}) = W^{m,n}$. According to (2.11), the coproduct is given by

$$\begin{aligned} \Delta(\partial_{t_p}) &= 1 \otimes \partial_{t_p} + \partial_{t_p} \otimes 1 \,, & (5.10) \\ \Delta(\partial_{\theta_{m,n}}) &= 1 \otimes \partial_{\theta_{m,n}} + \partial_{\theta_{m,n}} \otimes 1 + \tfrac{1}{2}\big(\partial_{t_m} \otimes \partial_{t_n} - \partial_{t_n} \otimes \partial_{t_m}\big) \,. & (5.11) \end{aligned}$$

4. One may think of iterating the deformation and extension procedure by introducing new deformation parameters $\theta_{p,m,n}$ and $\theta_{m,n,r,s}$ with the pairs of extended hierarchy parameters $t_p, \theta_{m,n}$ and $\theta_{m,n}, \theta_{r,s}$, respectively, and so forth. Of course the product has then to be changed appropriately in order to preserve associativity. But this can indeed be done.

[13]) More generally, ψ could be taken to be an element of a left $\mathcal{A}$–module. This would require slight changes of the formalism.

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