Saturation and non-linear effects in diffractive processes

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Through a direct implementation of the saturation regime resulting from the unitarity limit in the impact parameter representation, we explore various possibilities for the energy dependence of hadronic scattering. We show that it is possible to obtain a good description of the scattering amplitude from a hard pomeron provided one includes non-linear effects.

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1 Introduction

The most important results on the energy dependence of diffractive hadronic scattering were obtained from first principles (analyticity, unitarity and Lorentz invariance), which lead to specific analytic forms for the scattering amplitude as a function of its kinematical parameters — s, t, and u. Analytic S–matrix theory relates the high-energy behaviour of hadronic scattering to the singularities of the scattering amplitude in the complex angular momentum plane. One of the important theorems is the Froissart–Martin bound [1] which states that the high–energy cross section for the scattering of hadrons is limited by

$$\sigma_{\rm tot}^{\rm max} = \frac{2\pi}{\mu^2} \log^2\left(\frac{s}{s_0}\right), \tag{1}$$

where s_0 is a scale factor and μ the lightest hadron mass (*i.e.* the pion mass). As the coefficient in front of the logarithm is very large, "saturation of the Froissart–Martin bound" usually refers to an energy dependence of the total cross section rising as $\log^2 s$ rather than to a total cross section equal to (1).

Experimental data reveal that total cross sections grow with energy. This means that the leading contribution in the high–energy limit is given by the rightmost singularity in the complex–j plane, the pomeron, with intercept exceeding unity. In the framework of perturbative QCD, the intercept is also expected to exceed unity by an amount proportional to α_s [2]. At leading–log, one obtains a leading singularity at $J - 1 = 12 \log 2(\alpha_s/\pi)$. In this case, the Froissart–Martin bound is soon violated.

But this is not the whole story, as there is another important consequence of unitarity, which is connected with the impact–parameter representation.

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2 Unitarity bound

Unitarity of the scattering matrix $SS^{+}=1$ is connected with the properties of the scattering amplitude in the impact parameter representation as it is equivalent at high energy to a decomposition in partial–wave amplitudes. As energy increases, the scattering amplitude in impact parameter can saturate the unitarity bound for some impact parameter b_s.

To satisfy the unitarity condition, there are different prescriptions. Two of them are based on particular solutions of the unitarity equation.

For two–particle elastic scattering, the latter can be written

$$\Im\langle p_1,p_2,\mathrm{out}|T|p_1,p_2,\mathrm{in}\rangle = \frac{(2\pi)^4}{2}\sum_{\gamma}\int \mathrm{d}\gamma\,\delta\left(\sum_{r=1}^{2}p_r-\sum_{r=1}^{n}q_r\right)|T_{\gamma\alpha}|^2. \quad (2)$$

The scattering amplitude in the impact parameter representation is defined as

$$T(s,t)=\int_0^{\infty} b\,\mathrm{d}b J_0(b\Delta) f(b,s)\,. \quad (3)$$

with

$$\Im f(b,s)\leq 1\,. \quad (4)$$

and

$$\Im f(s,b)=\big[\Im f(s,b)\big]^2+\big[\Re f(s,b)\big]^2. \quad (5)$$

One of the possibilities is obtained in the U–matrix approach [4]:

$$f(s,b)=\frac{U(s,b)}{1-\mathrm{i}U(s,b)}\,. \quad (6)$$

This solution leads to a nonstandard behaviour of the ratio [4]

$$\frac{\sigma_{\mathrm{el}}}{\sigma_{\mathrm{tot}}}\ \rightarrow\ 1\,, \quad (7)$$

as $s\rightarrow\infty$. For the highest energies reached so far ($\sqrt{s}=2.0\,\mathrm{TeV}$), such a ratio is 0.25. It is not too far from the standard value 1/2, but it is very far from the solution in the U–matrix representation.

The second possible solution of the unitarity condition corresponds to the eikonal representation

$$T(s,t)=\int_0^{\infty} b\,\mathrm{d}b J_0(b\Delta)\big(1-\exp(-\chi(s,b)\big)\,. \quad (8)$$

with $t=\Delta^2$. If one takes the eikonal phase in factorized form

$$\chi(s,b)=h(s)\,f(b)\,, \quad (9)$$

one usually supposes that, despite the fact that the energy dependence of $h(s)$ can be a power

$$h(s)\sim s^{\Delta}\,, \quad (10)$$

the total cross section will satisfy the Froissart bound

$$\sigma_{\rm tot} \leq a \log^2 s \,. \qquad (11)$$

We find in fact that the energy dependence of the imaginary part of the amplitude and hence of the total cross section depends on the form of $f(b)$, *i.e.* on the s and t dependence of the slope of the elastic scattering amplitude. If $f(b)$ decreases as a power of b, the Froissart–Martin bound will always be violated. In the case of other forms of the b dependence, a special analysis [5] is required. Note that the eikonal form does not correspond to a saturation of the amplitude: in this case, $\Im f(s,b)$ reaches the black disk limit only asymptotically. Hence, the saturation of the black disk limit and the eikonal representation lead to different results for the scattering amplitude in the momentum transfer representation.

Let us take, as an example, the hard plus soft pomeron model [6, 7] which includes two simple poles (a soft and a hard pomeron) to describe pp and $\bar{p}p$ scattering. In this case, the pp–elastic scattering amplitude is proportional to the hadron form factors and can be approximated at small t by:

$$\Im T^0(s,t) \approx \left[h_1 (s/s_0)^{\epsilon_1} {\rm e}^{\alpha_1' t \log(s/s_0)} + h_2 (s/s_0)^{\epsilon_2} {\rm e}^{\alpha_2' t \log(s/s_0)} \right] F^2(t) \,. \qquad (12)$$

where $h_1 = 4.7$ and $h_2 = 0.005$ are the coupling of the "soft" and "hard" pomerons, and $\epsilon_1 = 0.0072$, $\alpha_1' = 0.25$, and $\epsilon_2 = 0.45$, $\alpha_2' = 0.20$ are the intercepts and the slopes of the two pomeron trajectories. The normalization s_0 will be dropped below and s contains implicitly the phase factor $\exp(-{\rm i}\pi/2)$. $F^2(t)$ is the square of the Dirac elastic form factor, which can be approximated by the sum of three exponentials [8]. We then obtain in the impact parameter representation a specific form for the profile function $\Gamma(b,s)$ [5], which we show in Fig. 1. One can see that at some energy and at small b, $\Gamma(b,s)$ reaches the black disk limit. For one–pomeron exchanges, this will be in the region $\sqrt{s} \approx 1.5\,{\rm TeV}$. If one adds to the model 2–pomeron exchanges, the resulting Γ_2 will saturate at $\sqrt{s} = 4.5\,{\rm TeV}$.

Saturation of the profile function will control the behaviour of $\sigma_{\rm tot}$ at super–high energies. Note that one cannot simply cut the profile function sharply as this would lead to a non-analytic amplitude, and to specific diffractive patterns in the total cross section and in the slope of the differential cross sections. Furthermore, we have to match at large impact parameter the behaviour of the unsaturated profile function. We have tried some specific matching patterns which softly interpolate between both regimes. The interpolating functions give unity in the large impact parameter region and force the profile function to approach the saturation scale b_s as a Gaussian.

If we take a single simple pole for the scattering amplitude, with an exponential form factor, the radius of saturation, and its dependence on energy, can be obtained analytically:

$$R(s)^2 \sim 4 \left[B \log\left(\frac{h}{2B}\right) + \left(B + \log\frac{h}{2(B+\epsilon\log s)}\right) \epsilon \log s + \epsilon^2 \log^2 s \right],$$

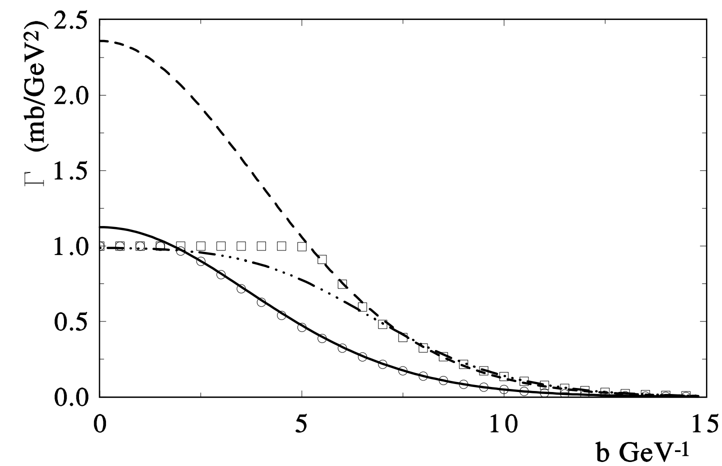

Fig. 1. The profile function of proton–proton scattering. (hard line and circles – at $\sqrt{s} = 2\,\mathrm{TeV}$ without and with saturation; dashed line and squares – at $\sqrt{s} = 14\,\mathrm{TeV}$ without and with saturation; dash-dotted line – the eikonal form (13) at $\sqrt{s} = 14\,\mathrm{TeV}$)

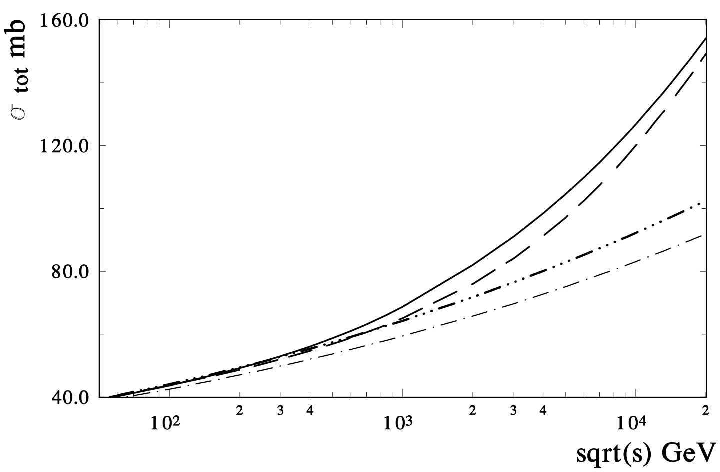

Fig. 2. σ_{tot} calculated in different approaches. (The hard line – saturation regime; the long-dashed line – with the eikonal representation (13); the dash-dot-dotted line – with soft pomeron and unitarized hard pomeron [7]; the dash-dotted line – with only a soft pomeron)

where B is the average slope at small t. Hence the total cross section grows logarithmically at medium energies and grows like $\log^2 s$ at very high energies.

We show in Fig. 2 the possible behaviours of the total cross section at very high energies, depending on the model and on the unitarization scheme. We also show there the result of a simple eikonalization, where we took

$$T_{pp}(s,t=0) \;=\; 2\int \mathrm{d}^2 b\,\left[1-\exp\bigl(h_{\rm eik}G(s,b)\bigr)\right]. \tag{13}$$

and where $h_{\rm eik}=1.2$ was chosen so that the values $\sigma_{\rm tot}$ determined by (13) and by the saturation procedure are equal at $\sqrt{s}=50\,\mathrm{GeV}$, and $G(s,b)$ is the Fourier transform of $\Im T^0(s,t)$ given in (12).

3 Non-linear equations

The problem of the implementation of unitarity via saturation is that the matching procedure seems arbitrary. Hence we considered a different approach to saturate the amplitude. It is connected with the non-linear saturation processes which have been considered in a perturbative QCD context [9, 10]. Such processes lead to an infinite set of coupled evolution equations in energy for the correlation functions of multiple Wilson lines [11]. In the approximation where the correlation functions for more than two Wilson lines factorize, the problem reduces to the non-linear Balitsky–Kovchegov (BK) equation [11, 12].

It is unclear how to extend these results to the non-perturbative region, but one will probably obtain a similar equation. In fact we found simple differential equations that reproduce either the U–matrix or the eikonal representation. We can first consider saturation equations of the general form

$$\frac{\partial N(\xi,b)}{\partial\xi}=\mathcal{S}(N)\,, \tag{14}$$

with N the true (saturated) imaginary part of the amplitude. We shall impose the following conditions:

(a) $N\to 1$ as $s\to\infty$,

(b) $\partial N/\partial\xi\to 0$ as $s\to\infty$,

(c) $\mathcal{S}(N)$ has a Taylor expansion in N, and considering the first term only gives the hard pomeron $N_{\rm bare}=f(b)s^{\Delta}$. Similarly, we fix the integration constant by demanding that the first term of the expansion in s^{Δ} reduces to $N_{\rm bare}$.

Inspired by the BK results, we shall use the evolution variable $\xi=\log s$. If we want to fulfil condition (c), then we need to take $\mathcal{S}(N)=\Delta N+O(N^2)$. Conditions (a) and (b) then give $\mathcal{S}(N)=\Delta(N-N^2)$ as the simplest saturating function. The resulting equation

$$\frac{\partial N}{\partial\log s}=\Delta(N-N^2) \tag{15}$$

has the solution

$$N=\frac{f(b)s^{\Delta}}{f(b)s^{\Delta}+1}\,. \tag{16}$$

One can in fact go further: eq. (15) has been written for the imaginary part of the amplitude. If we want to generalize it to a complex amplitude, so that it reduces to (15) when the real part vanishes, we must take:

$$\frac{\partial A}{\partial \log s} = \Delta(A + \mathrm{i}A^2)\,. \tag{17}$$

The solution of this is exactly the form (6) obtained in the U–matrix formalism, for $\Im U(s,b) = s^{\Delta} f(b)$.

Many other unitarization schemes are possible, depending on the function $\mathcal{F}(N)$. We shall simply indicate here that the eikonal scheme can be obtained as follows:

$$\frac{\partial N}{\partial \log s} = \Delta(1-N)\big(-\log(1-N)\big)\,. \tag{18}$$

Other unitarization equations can be easily obtained via another first–order equation. The idea here is that the saturation variable is the imaginary part of the bare amplitude. One can then write

$$\frac{\partial N}{\partial N_{\mathrm{bare}}} = \mathcal{F}'(N) \Rightarrow \frac{\partial N}{\partial \log s} = \frac{\partial N_{\mathrm{bare}}}{\partial \log s}\mathcal{F}'(N)\,, \tag{19}$$

with N_{bare} the unsaturated amplitude.

This will trivially obey the conditions (a) − (c) above, and saturate at $N = 1$. Choosing $\mathcal{F}'(N) = 1 - N$ gives the eikonal solution

$$N(b,s) = 1 - \exp\big(-N_{\mathrm{bare}}(b,s)\big)\,, \tag{20}$$

whereas $\mathcal{F}'(N) = (1-N)^2$ leads to the U–matrix representation (16).

We can come to the same results if we solve this equation via an iteration procedure. For that let us take some model of the hadron interaction in which the main hadron–hadron interaction is created by the valence quarks surrounded by clouds of sea quarks.

Ref. [13] proposed a picture in which the whole impact parameter interval is divided into small and large distances: the BFKL approximation works within a small radius R_0 and the BK representation works for large impact parameters.

We examine two cases: the first includes only a short–range form factor (as a simple Gaussian), and the second uses a short–range form factor and a long–range one (in the form of a MacDonald function).

So, we divide the whole energy interval in small pieces inversely proportional to $s^{2\Delta}$. We then obtain the step in s

$$s_0 = s^{\Delta}/s^{2\Delta} = 1/s^{\Delta} \tag{21}$$

and the number of such pieces will be

$$n = \mathrm{int}\,(s^{2\Delta})\,, \tag{22}$$

where the function int chooses the nearest natural number.

$N(s,b)$ after this small energy interval will be

$$N_0(s_0,b) = s_0 f(b) = f(b)/s^{\Delta}\,. \tag{23}$$

We calculate the derivative δ on this small interval and increase $N(s_0,b)$ by $\delta N_0(s_0,b)$:

$$\begin{aligned} \delta_i &= 1 - N_i\,, \\ N_{i+1} &= N_i + \delta\, N_0(s_0,b)\,. \end{aligned} \tag{24}$$

We then iterate the above procedure for $N_i = N(s_i,b)$ until we get to the end of the interval. It is clearly understood that, if the energy is sufficiently high, we obtain for some iteration $N_k = 1$ and $\delta = 0$, so we reach the saturation bound at some impact parameter. Of course this energy will depend on the form of $f(b)$.

In this case

$$T(s,t=0) \;=\; \int_0^{\infty} b\,\mathrm{d}b\,N(s,b) \tag{25}$$

and

$$\sigma_{\mathrm{tot}}(s) = 4\pi\;\Im T(s,t=0)\,. \tag{26}$$

The results of our calculations are shown in Fig. 3. We can see that energy dependence of the total cross section is a simple logarithm. So, in this case, $\sigma_{\mathrm{tot}}(s)$ does not saturate the Froissart unitarity bound and coincides with the eikonal solution with a Gaussian eikonal.

Note that we obtain in this case the right energy dependence not only for $\sigma_{\mathrm{tot}}(s)$, but also for the differential cross sections. Of course, this picture is only qualitative.

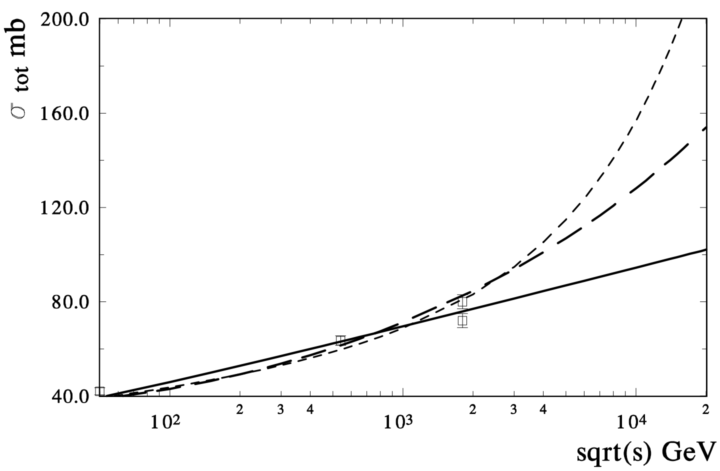

Fig. 3. σ_{tot} calculated in different approaches. (The hard line – $f(b)$ in Gaussian form; the long-dashed line – with the contribution of the large distances; the dashed line – in the soft + hard pomeron model)

For a quantitative description of the different features of the diffraction processes, we presumably need to take into account many different effects and a more complicated structure for the non-linear equation.

We have shown that the most usual unitarization schemes could be recast into differential equations which are reminiscent of saturation equations [11, 12]. Such an approach can be used to build new unitarization schemes and may also shed some light on the physical processes underlying the saturation regime.

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