On systems of diffusion equations

C. Sophocleous

Department of Mathematics and Statistics, University of Cyprus, CY 1678 Nicosia, Cyprus

R.J. Wiltshire

The Division of Mathematics and Statistics, The University of Glamorgan, Pontypridd CF37 1DL, Great Britain

We consider systems of diffusion equations that have considerable interest in Soil Science and Mathematical Biology. We construct non-local symmetries, known as potential symmetries. Furthermore we present linearizing mappings.

PACS: 02.30.Jr; 02.20.Tw
Key words: diffusion equations, potential symmetries, linearizing mappings

1 Introduction

Whilst systems of pure diffusion equations, in both their linear and nonlinear forms are well known and have many physical and biological applications, the research described here focusses on less familiar cases where diffusion coefficients or other shape functions are defined either in general or poor analytic terms. Of particular interest here is the case of the extension of Richards equation, which describes the movement of water in a homogeneous unsaturated soil, to cases describing the combined transport of water vapour and heat under a combination of gradients of soil temperature and volumetric water content. Such coupled transport is of considerable significance in semi–arid environments where moisture transport often occurs essentially in the water vapour phase [1]. Under these conditions the transport equations, valid in a vertical column of soil, may be written in the form [2]

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}\left[f(u,v)\frac{\partial u}{\partial x} + g(u,v)\frac{\partial v}{\partial x}\right], \\ \frac{\partial v}{\partial t} &= \frac{\partial}{\partial x}\left[h(u,v)\frac{\partial u}{\partial x} + k(u,v)\frac{\partial v}{\partial x}\right], \end{aligned} \tag{1}$$

where $u(x,t)$ and $v(x,t)$ are respectively the soil temperature and volumetric water content at depth x and time t. It is important to realize that extensions which include the coupled diffusion of solute follow in an obvious way.

Lie group methods are perhaps the most powerful currently available in finding exact solutions of nonlinear partial differential equations (pdes). Probably the most useful method is the applications of Lie point transformations which are those that form a continuous Lie group of transformations, which leave the pde invariant. Symmetries of this pde are then revealed, perhaps enabling new solutions to be found directly or via similarity reductions. The classical method of finding Lie symmetries is first to find infinitesimal transformations, with the benefit of linearization, and then to extend these to groups of finite transformations. This method is easy to apply and well established in the last few years [3, 4]. The problem of finding Lie symmetries of system (1) has been considered elsewhere [5, 6].

Bluman et al [3, 7] introduced a method for finding a new class of symmetries for a system of partial differential equations $\Delta(x,u)$, in the case that at least one of the pdes can be written in conserved form. If we introduce potential variables v for the pdes written in conserved forms as further unknown functions, we obtain a system $Z(x,u,v)$. Any Lie group of transformations for $Z(x,u,v)$ induces a symmetry for $\Delta(x,u)$. When at least one of the infinitesimals which correspond to the variables x and u depends explicitly on the potential v, then the local symmetry of $Z(x,u,v)$ induces a nonlocal symmetry of $\Delta(x,u)$. These nonlocal symmetries are called *potential symmetries*. More details about potential symmetries and their uses can be found in [3, 8]. The potential symmetries of Richard's equation have been discussed in [9].

The purpose of the present paper is to derive potential symmetries for the system (1). That is, to determine functional forms of $f(u,v)$, $g(u,v)$, $h(u,v)$ and $k(u,v)$ for which the system admits such nonlocal symmetries. We note that both equations of the system (1) are written in conserved form. We can therefore either introduce one potential variable and write one equation as a system of two equations and keep the other equation in the same form, or introduce two potential variables and write system (1) as a system of four equations. In particular, if we introduce the potential variable w system (1) can be written in the form

$$\begin{aligned} w_x &= u\,, \\ w_t &= f(u,v)u_x + g(u,v)v_x\,, \\ v_t &= \left[h(u,v)u_x + k(u,v)v_x\right]_x\,. \end{aligned} \tag{2}$$

If we introduce potential variables w and z, system (1) can be written as a system of four equations,

$$\begin{aligned} w_x &= u\,, \\ w_t &= f(u,v)u_x + g(u,v)v_x\,, \\ z_x &= v\,, \\ z_t &= h(u,v)u_x + k(u,v)v_x\,. \end{aligned} \tag{3}$$

We derive Lie point symmetries for each of the above systems. However we shall only be interested to those symmetries that produce nonlocal (potential) symmetries for the original system (1).

System (1) can be seen as a generalization of the well known nonlinear heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K(u)\frac{\partial u}{\partial x}\right). \tag{4}$$

Equation (4) have been exhaustively studied from the point of group analysis. A summary of these results can be found in [10]. Lie symmetries of (4) have been derived by Ovsiannikov [11] who proved that (i) if $K(u)$ is an arbitrary function then the equation admits a 3–parameter Lie group of transformations; (ii) if $K(u) = u^n$ or $K(u) = \mathrm{e}^u$, then it admits a 4–parameter Lie group of transformations; and (iii) if $K(u) = u^{-4/3}$, it admits a 5–parameter Lie group of transformations.

If we introduce the potential variable v we obtain the auxiliary system of (4)

$$\begin{aligned} \frac{\partial v}{\partial x} &= u\,, \\ \frac{\partial v}{\partial t} &= K(u)\frac{\partial u}{\partial x}\,. \end{aligned} \tag{5}$$

Bluman et. al have shown that the nonlinear heat conduction equation (4) admits a potential symmetry, corresponding to auxiliary system (5), if and only if the conductivity $K(u)$ is of the form

$$K(u) = \frac{1}{u^2 + pu + q}\ \exp\left[r\int \frac{\mathrm{d}u}{u^2 + pu + q}\right],$$

where p, q, and r are arbitrary constants. The most interesting case is when $K(u) = u^{-2}$, where equation (4) admits the following potential symmetries

$$\begin{aligned} \Gamma_1 &= -xv\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial v} + (uv + xu^2)\frac{\partial}{\partial u}\,, \\ \Gamma_2 &= -x(v^2 + 2t)\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} + 4vt\frac{\partial}{\partial v} + (4ut + uv^2 + 2ut + 2u^2vx)\frac{\partial}{\partial u}\,, \\ \Gamma_\infty &= h(t,v)\frac{\partial}{\partial x} - u^2\frac{\partial h(t,v)}{\partial v}\frac{\partial}{\partial u}\,, \end{aligned}$$

where $h(t,v)$ is an arbitrary solution of the linear heat equation

$$h_{vv} - h_t = 0\,.$$

In [3] it is shown that an invertible mapping that transforms a non-linear pde does not exist if the nonlinear pde does not admit an infinite–parameter Lie group of contact transformations. Also such mappings do not exist for a non-linear system of pdes if the system does not admit an infinite–parameter Lie group of transformations. If such infinite–parameter groups exist then the non-linear pde (or system of non-linear pdes) can be transformed into a linear pde (or into a system of linear pdes) provided that these groups satisfy certain criteria [3]. The above infinite–dimensional symmetry admitted by

$$\begin{aligned} v_x &= u\,, \\ v_t &= u^{-2}u_x\,, \end{aligned} \tag{6}$$

satisfies the required criteria. Hence, system (6) can be linearized. The procedure for determining invertible mappings with the employment of infinite–parameter

Lie groups of transformations is well explained in Ref. [3]. The above infinite–dimensional Lie symmetry leads to the mapping

$$t' = t\,, \quad x' = v\,, \quad v' = x\,, \quad u' = \frac{1}{u}\,, \tag{7}$$

that transforms any solution $(u'(x',t'), v'(x',t'))$ of the linear system of pdes

$$v'_{x'} = u'\,, \quad v'_{t'} = u'_{x'}\,, \tag{8}$$

to a solution $\big(u(x,t), v(x,t)\big)$ of the nonlinear system (6). In turn the mapping (7) produces the one-to-one contact transformation

$$\mathrm{d}x' = u\,\mathrm{d}x + u^{-2}u_x\,\mathrm{d}t\,, \quad \mathrm{d}t' = \mathrm{d}t\,, \quad u' = \frac{1}{u}\,,$$

which transforms the linear diffusion equation

$$\frac{\partial u'}{\partial t'} = \frac{\partial^2 u'}{\partial x'^2} \tag{9}$$

into the pde

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right). \tag{10}$$

Using the inverse mapping of (7)

$$t = t'\,, \quad x = v'\,, \quad v = x'\,, \quad u = \frac{1}{u'}\,,$$

we deduce the contact transformation

$$\mathrm{d}x = u'\mathrm{d}x' + u'_{x'}\mathrm{d}t'\,, \quad \mathrm{d}t = \mathrm{d}t'\,, \quad u = \frac{1}{u'}\,, \tag{11}$$

which also connects the linear diffusion equation (9) and the nonlinear pde (10).

Now using the transformation,

$$t' = t\,, \quad x' = v\,, \quad v' = x\,,$$

known as pure hodograph transformation, that maps the linear diffusion equation (9) into the pde

$$v_t = \frac{v_{xx}}{v_x^2}\,, \tag{12}$$

we deduce the one-to-one contact transformation

$$\mathrm{d}x' = u\,\mathrm{d}x + u^{-2}u_x\,\mathrm{d}t\,, \quad \mathrm{d}t' = \mathrm{d}t\,, \quad u' = x\,, \tag{13}$$

which transforms the linear diffusion equation (9) into the pde (10).

Taking the composition of the one-to-one contact transformations (11) and(13), we find the generalized auto–hodograph transformation [12]

$$\mathrm{d}x \mapsto xu\,\mathrm{d}x + \left(xu^{-2}u_x + u^{-1}\right)\mathrm{d}t\,, \quad \mathrm{d}t \mapsto \mathrm{d}t\,, \quad u \mapsto \frac{1}{x}$$

that leaves invariant the nonlinear equation (10).

The above results of equation (10) will be generalized to certain forms of the system (1) that admit infinite dimensional potential symmetries. In the next section we classify all forms of (1) that admit potential symmetries corresponding to the auxiliary system (2). Furthermore we present certain forms of (1) that admit potential symmetries corresponding to the auxiliary system (3). In section 3 we construct linearizing and hodograph–type transformations for special cases of (1). In the final section we present some ideas for further study in the spirit of the present paper.

2 Potential symmetries

2.1 First generation potential system

We consider the system (2) and infinitesimal transformations of the form

$$\begin{aligned}
x' &= x + \epsilon X(x,t,u,v,w) + O(\epsilon^2)\,,\\
t' &= t + \epsilon T(x,t,u,v,w) + O(\epsilon^2)\,,\\
u' &= u + \epsilon U(x,t,u,v,w) + O(\epsilon^2)\,,\\
v' &= v + \epsilon V(x,t,u,v,w) + O(\epsilon^2)\,,\\
w' &= w + \epsilon W(x,t,u,v,w) + O(\epsilon^2)\,.
\end{aligned} \tag{14}$$

We search for transformations of the above form that leave invariant the system (2). We only determine Lie symmetries of (2) that induce potential symmetries for the system (1). That is, we require that the infinitesimals of (14) satisfy the condition [3]

$$X_w^2 + T_w^2 + U_w^2 + V_w^2 \neq 0\,.$$

If this condition is not satisfied, then any symmetry of the form (14) projects into Lie point symmetry of (1).

Here we completely classified the forms of the system (1) that admit potential symmetries, corresponding to the auxiliary system (2). We omit any further analysis. The method for determining Lie symmetries is well known and can be found in a number of relevant books. See for example, Ref. [3, 4].

We present below the results. In each case we give the functional forms of $f(u,v)$, $g(u,v)$, $h(u,v)$ and $k(u,v)$ and the infinitesimal generator in the form

$$\Gamma = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v} + W\frac{\partial}{\partial w}\,.$$

We obtain five cases, where we only present those symmetries of (2) that induce potential symmetries of (1):

1. $f = \phi(uv^n)\,, \quad g = n\dfrac{u}{v}\,\phi(uv^n)\,, \quad h = \dfrac{v}{u}\,\phi(uv^n)\,, \quad k = n\phi(uv^n),$

$$\Gamma = ua'(w)\frac{\partial}{\partial u} - \frac{1}{n}\,va'(w)\frac{\partial}{\partial v} + a(w)\frac{\partial}{\partial w}\,,$$

where ϕ and a are arbitrary functions in their arguments. Here the prime denotes ordinary differentiation.

2. $f = uv^2\phi'(uv^2) - 2u^2v^4\psi'(uv^2) + \phi(uv^2) - 3uv^2\psi(uv^2),$
$g = 2\dfrac{u}{v}\left[uv^2\phi'(uv^2) - 2u^2v^4\psi'(uv^2) + \phi(uv^2) - 3uv^2\psi(uv^2)\right],$
$h = v^3\psi(uv^2),\ k = \phi(uv^2),$

$$\Gamma = 2uw\frac{\partial}{\partial u} - vw\frac{\partial}{\partial v} + w^2\frac{\partial}{\partial w}\,,$$

where ϕ is arbitrary function.

3. $f = \phi(u)\,, \quad g = \dfrac{\mu}{uv}\,, \quad h = \dfrac{v(2u^2\phi(u) + \mu)}{2u^3}\,, \quad k = \dfrac{\mu}{2u^2},$

$$\Gamma = vw\frac{\partial}{\partial v} + \mu t\frac{\partial}{\partial w}\,,$$

where ϕ is arbitrary function and μ is a constant.

The following case is a special case of 3. Here the system (1) admits two potential symmetries.

4. $f = \dfrac{1}{u^2}\,, \quad g = \dfrac{\mu}{uv}\,, \quad h = \dfrac{(\mu+2)v}{2u^3}\,, \quad k = \dfrac{\mu}{2u^2},$

$$\Gamma_1 = vw\frac{\partial}{\partial v} + \mu t\frac{\partial}{\partial w}\,,$$
$$\Gamma_2 = t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u} + \frac{v}{2\mu}(\mu t + w^2)\frac{\partial}{\partial v} + tw\frac{\partial}{\partial w}\,.$$

5. $f = \dfrac{1}{u}\,, \quad g = \dfrac{\mu}{v}\,, \quad h = \dfrac{v}{u^2}\,, \quad k = \dfrac{\mu}{u},$

$$\Gamma = vw\frac{\partial}{\partial v}\,.$$

2.2 Second generation potential system

In this section we determine potential symmetries for the system (1) that correspond to auxiliary system (3). Hence, we consider infinitesimal transformations of the form

$$
\begin{aligned}
x' &= x + \epsilon X(x,t,u,v,w,z) + O(\epsilon^2)\,, \\
t' &= t + \epsilon T(x,t,u,v,w,z) + O(\epsilon^2)\,, \\
u' &= u + \epsilon U(x,t,u,v,w,z) + O(\epsilon^2)\,, \\
v' &= v + \epsilon V(x,t,u,v,w,z) + O(\epsilon^2)\,, \\
w' &= w + \epsilon W(x,t,u,v,w,z) + O(\epsilon^2)\,, \\
z' &= z + \epsilon Z(x,t,u,v,w,z) + O(\epsilon^2)\,.
\end{aligned}
\tag{15}
$$

Unlike in the previous subsection where we classified the forms of (1) that admit potential symmetries, here we present a number of examples. That is, we give certain special cases of (1) that admit potential symmetries. The task of a complete classification of potential symmetries for the system (1) corresponding to the auxiliary system (3) remains an open problem.

The symmetry group generator has the form:

$$
\Gamma = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v} + W\frac{\partial}{\partial w} + Z\frac{\partial}{\partial z}\,. \tag{16}
$$

Initial symmetry analysis gives:

$$
\begin{aligned}
X &= X\,(x,t,w,z)\,, \\
T &= T\,(t)\,, \\
W &= W\,(x,t,w,z)\,, \\
Z &= Z\,(x,t,w,z)
\end{aligned}
$$

and:

$$
\begin{aligned}
U &= U\,(x,t,u,v,w,z) = vW_z + uW_w + W_x - uvX_z - u^2X_w - uX_x\,, \\
V &= V\,(x,t,u,v,w,z) = vZ_z + uZ_w + Z_x - v^2X_z - uvX_w - vX_x\,.
\end{aligned}
$$

We give six examples. For each example we state the functional forms of $f(u,v)$, $g(u,v)$, $h(u,v)$ and $k(u,v)$ and the independent generators.

Example 1:

$$
f = \frac{u}{v^2}\,, \quad g = -\frac{u^2}{v^3}\,, \quad h = \frac{1}{v}\,, \quad k = -\frac{u}{v^2}\,.
$$

The independent generators are:

$$
\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial t}\,, \\
\Gamma_2 &= x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\,, \\
\Gamma_3 &= x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}\,, \\
\Gamma_4 &= -\left(twa_{zzz} + 3xa_z\right)\frac{\partial}{\partial x} +
\end{aligned}
$$

$$
\begin{aligned}
&+\left(uvtwa_{zzzz}+u^2ta_{zzz}+(vw+3xuv)\,a_{zz}+4ua_z\right)\frac{\partial}{\partial u}+\\
&+\left(v^2twa_{zzzz}+uvta_{zzz}+3xv^2a_{zz}+5va_z\right)\frac{\partial}{\partial v}+\\
&+wa_z\frac{\partial}{\partial w}+2a\frac{\partial}{\partial z}\,,\\
\Gamma_5 &= q\frac{\partial}{\partial x}-\left(uvq_z+u^2q_w\right)\frac{\partial}{\partial u}-\left(v^2q_z+uvq_w\right)\frac{\partial}{\partial v}\,,\\
\Gamma_6 &= -ts_{zz}\frac{\partial}{\partial x}+\left(uvts_{zzz}+vs_z\right)\frac{\partial}{\partial u}+v^2ts_{zzz}\frac{\partial}{\partial v}+s\frac{\partial}{\partial w}\,,
\end{aligned}
$$

where $q=q\,(w,z)$, $a=a\,(z)$ and $s=s\,(z)$ are arbitrary functions. Symmetries $\Gamma_1-\Gamma_3$ project into point symmetries of (1), while $\Gamma_4-\Gamma_6$ induce potential symmetries of (1).

Example 2:

$$
f=\frac{uv}{(u+v)^3}\,,\quad g=-\frac{u^2}{(u+v)^3}\,,\quad h=\frac{v^2}{(u+v)^3}\,,\quad k=-\frac{uv}{(u+v)^3}\,.
$$

The independent generators are:

$$
\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial t}\,,\\
\Gamma_2 &= \frac{\partial}{\partial z}\,,\\
\Gamma_3 &= -2x\frac{\partial}{\partial x}+2u\frac{\partial}{\partial u}+(u+3v)\frac{\partial}{\partial v}+\xi\frac{\partial}{\partial z}\,,\\
\Gamma_4 &= x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}-u\frac{\partial}{\partial u}-v\frac{\partial}{\partial v}\,,\\
\Gamma_5 &= ta_{\xi\xi}\frac{\partial}{\partial x}-(u+v)\left(a_\xi+uta_{\xi\xi\xi}\right)\frac{\partial}{\partial u}+\\
&\quad+(u+v)\left(a_\xi-vta_{\xi\xi\xi}\right)\frac{\partial}{\partial v}-a\frac{\partial}{\partial w}+a\frac{\partial}{\partial z}\,,\\
\Gamma_6 &= q\frac{\partial}{\partial x}-\left(uvq_z+u^2q_w\right)\frac{\partial}{\partial u}-\left(v^2q_z+uvq_w\right)\frac{\partial}{\partial v}\,,\\
\Gamma_7 &= -\left(3xb_\xi+wtb_{\xi\xi\xi}\right)\frac{\partial}{\partial x}+\\
&\quad+\left[u\,(u+v)\,wtb_{\xi\xi\xi\xi}+u^2tb_{\xi\xi\xi}+(w+3ux)\,(u+v)\,b_{\xi\xi}+4ub_\xi\right]\frac{\partial}{\partial u}+\\
&\quad+\left[v\,(u+v)\,wtb_{\xi\xi\xi\xi}+uvtb_{\xi\xi\xi}+(-w+3vx)\,(u+v)\,b_{\xi\xi}+(u+5v)\,b_\xi\right]\frac{\partial}{\partial v}+\\
&\quad+wb_\xi\frac{\partial}{\partial w}+\left(2b-wb_\xi\right)\frac{\partial}{\partial z}\,,
\end{aligned}
$$

where $q=q\,(w,z)$, $a=a\,(\xi)$ and $b=b\,(\xi)$ are arbitrary functions and $\xi=w+z$. Here symmetries $\Gamma_1-\Gamma_4$ project into point symmetries of (1), while $\Gamma_5-\Gamma_7$ induce potential symmetries of (1).

Example 3:

$$f=\frac{1}{(u+v)^2}\,,\quad g=0\,,\quad h=0\,,\quad k=\frac{1}{(u+v)^2}\,.$$

The independent generators are:

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial t}\,,\\
\Gamma_2 &= \frac{\partial}{\partial w}\,,\\
\Gamma_3 &= -\frac{\partial}{\partial u}+\frac{\partial}{\partial v}-x\frac{\partial}{\partial w}+x\frac{\partial}{\partial z}\,,\\
\Gamma_4 &= -x\frac{\partial}{\partial x}+(u+v)\,\frac{\partial}{\partial v}-w\frac{\partial}{\partial w}+w\frac{\partial}{\partial z}\,,\\
\Gamma_5 &= 2t\frac{\partial}{\partial t}+(u+v)\,\frac{\partial}{\partial u}+(w+z)\,\frac{\partial}{\partial w}\,,\\
\Gamma_6 &= -x\left(t+\frac{\xi^2}{2}\right)\frac{\partial}{\partial x}+2t^2\frac{\partial}{\partial t}+\xi\,(u+v)\,(xu-w)\,\frac{\partial}{\partial u}+\\
&\quad+(u+v)\left(\frac{\xi^2}{2}+w\xi+3t+xv\xi\right)\frac{\partial}{\partial v}-\\
&\quad-w\left(t+\frac{\xi^2}{2}\right)\frac{\partial}{\partial w}+\left(2tz+3wt+\frac{w\xi^2}{2}\right)\frac{\partial}{\partial z}\,,\\
\Gamma_7 &= -\,(u+v)\,b_\xi\frac{\partial}{\partial u}+(u+v)\,b_\xi\frac{\partial}{\partial v}-b\frac{\partial}{\partial w}+b\frac{\partial}{\partial z}\,,\\
\Gamma_8 &= a\frac{\partial}{\partial x}-u\,(u+v)\,a_\xi\frac{\partial}{\partial u}-v\,(u+v)\,a_\xi\frac{\partial}{\partial v}\,,\\
\Gamma_9 &= -u\frac{\partial}{\partial u}+u\frac{\partial}{\partial v}-w\frac{\partial}{\partial w}+w\frac{\partial}{\partial z}\,,\\
\Gamma_{10} &= -x(w+z)\frac{\partial}{\partial x}+(u+v)(ux-w)\frac{\partial}{\partial u}+(u+v)(vx+2w+z)\frac{\partial}{\partial v}-\\
&\quad-w(w+z)\frac{\partial}{\partial w}+[2t+w(w+z)]\frac{\partial}{\partial z}\,,\\
\Gamma_{11} &= w\frac{\partial}{\partial x}+u^2\frac{\partial}{\partial u}-uv\frac{\partial}{\partial v}\,,
\end{aligned}$$

where $a=a\,(\xi,t)$ and $b=b\,(\xi,t)$, $\xi=w+z$, satisfy the linear heat equations $a_{\xi\xi}=a_t$, $b_{\xi\xi}=b_t$. Here symmetries $\Gamma_1-\Gamma_5$ project into point symmetries of (1), while $\Gamma_6-\Gamma_{11}$ induce potential symmetries of (1).

Example 4:

$$f=\frac{u}{(u+v)^3}\,,\quad g=\frac{u}{(u+v)^3}\,,\quad h=\frac{v}{(u+v)^3}\,,\quad k=\frac{v}{(u+v)^3}\,.$$

The independent generators are:

$$
\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial t}\,,\\
\Gamma_2 &= \frac{\partial}{\partial w}\,,\\
\Gamma_3 &= x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\,,\\
\Gamma_4 &= 2t\frac{\partial}{\partial t} + (u+v)\,\frac{\partial}{\partial u} + \xi\frac{\partial}{\partial w}\,,\\
\Gamma_5 &= a\frac{\partial}{\partial x} - u\,(u+v)\,a_\xi\frac{\partial}{\partial u} - v\,(u+v)\,a_\xi\frac{\partial}{\partial v}\,,\\
\Gamma_6 &= -\,(vq_z + uq_w)\,\frac{\partial}{\partial u} + (vq_z + uq_w)\,\frac{\partial}{\partial v} - q\frac{\partial}{\partial w} + q\frac{\partial}{\partial z}\,,
\end{aligned}
$$

where $q = q\,(w,z)$ is an arbitrary function and $a = a(t,\xi)$, $\xi = w + z$, satisfies $a_{\xi\xi} = a_t$. Here symmetries $\Gamma_1 - \Gamma_4$ project into point symmetries of (1), while $\Gamma_5 - \Gamma_6$ induce potential symmetries of (1).

Example 5:

$$
f = \frac{u}{v^2}\,,\quad g = \frac{\mu - u^2}{v^3}\,,\quad h = \frac{1}{v}\,,\quad k = -\frac{u}{v^2}\,.
$$

The independent generators are:

$$
\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial t}\,,\\
\Gamma_2 &= \frac{\partial}{\partial z}\,,\\
\Gamma_3 &= x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}\,,\\
\Gamma_4 &= 2t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v} + z\frac{\partial}{\partial z}\,,\\
\Gamma_5 &= w\frac{\partial}{\partial x} + (\mu - u^2)\frac{\partial}{\partial u} - uv\frac{\partial}{\partial v} + \mu x\frac{\partial}{\partial w}\,,\\
\Gamma_6 &= wz\frac{\partial}{\partial x} + [\mu(vx+z) - u(uz+vw)]\,\frac{\partial}{\partial u} - v(uz+vw)\frac{\partial}{\partial v} + \mu xz\frac{\partial}{\partial w} + 2\mu t\frac{\partial}{\partial z}\,,\\
\Gamma_7 &= (2\mu xt - wz^2)\frac{\partial}{\partial x} - 4\mu t^2\frac{\partial}{\partial t} + z(u^2z + 2uvw - 2\mu vx - \mu z)\frac{\partial}{\partial u} +\\
&\quad + v(-6\mu t + uz^2 + 2vwz)\frac{\partial}{\partial v} + \mu(2tw - xz^2)\frac{\partial}{\partial w} - 4\mu tz\frac{\partial}{\partial z}\,,\\
\Gamma_8 &= a\frac{\partial}{\partial x} - uva_z\frac{\partial}{\partial u} - v^2a_z\frac{\partial}{\partial v}\,,\\
\Gamma_9 &= vb_z\frac{\partial}{\partial u} + b\frac{\partial}{\partial w}\,,
\end{aligned}
$$

where $a = a(t,z)$, $b = b(t,z)$ satisfy $a_t + b_{zz} = 0$ and $b_t + \mu a_{zz} = 0$. Here symmetries $\Gamma_1 - \Gamma_4$ project into point symmetries of (1), while $\Gamma_5 - \Gamma_9$ induce potential symmetries of (1).

Example 6:

$$f = \frac{uv+\mu}{(u+v)^3}, \quad g = \frac{\mu - u^2}{(u+v)^3}, \quad h = \frac{v^2-\mu}{(u+v)^3}, \quad k = -\frac{uv+\mu}{(u+v)^3}.$$

The independent generators are:

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial t}, \\
\Gamma_2 &= \frac{\partial}{\partial z}, \\
\Gamma_3 &= x\frac{\partial}{\partial x} - (u+v)\frac{\partial}{\partial v} + w\frac{\partial}{\partial w} - w\frac{\partial}{\partial z}, \\
\Gamma_4 &= 2t\frac{\partial}{\partial t} + (u+v)\frac{\partial}{\partial v} + (w+z)\frac{\partial}{\partial z}, \\
\Gamma_5 &= w\frac{\partial}{\partial x} + (\mu - u^2)\frac{\partial}{\partial u} - (\mu + uv)\frac{\partial}{\partial v} + \mu x\frac{\partial}{\partial w} - \mu x\frac{\partial}{\partial z}, \\
\Gamma_6 &= w(w+z)\frac{\partial}{\partial x} + (\mu ux + \mu vx + \mu w + \mu z - 2u^2 w - u^2 z - uvw)\frac{\partial}{\partial u} - \\
&\quad -(\mu ux + \mu vx + \mu w + \mu z + 2uvw + uvz + v^2 w)\frac{\partial}{\partial v} + \\
&\quad +\mu x(w+z)\frac{\partial}{\partial w} + \mu(2t - wx - xz)\frac{\partial}{\partial z}, \\
\Gamma_7 &= \left[w(w+z)^2 - 2\mu xt\right]\frac{\partial}{\partial x} + 4\mu t^2\frac{\partial}{\partial t} + \\
&\quad + \left[2\mu x(u+v)(w+z) + \mu(w+z)^2 - u^2(3w^2 + 4wz + z^2) - 2uvw(w+z)\right]\frac{\partial}{\partial u} + \\
&\quad +\big[6\mu t(u+v) - 2\mu x(u+v)(w+z) - \mu(w+z)^2 - \\
&\qquad\qquad -uv(3w^2 + 4wz + z^2) - 2v^2 w(w+z)\big]\frac{\partial}{\partial v} + \\
&\quad +\mu\left[x(w+z)^2 - 2tw\right]\frac{\partial}{\partial w} + \mu\left[2t(3w+2z) - x(w+z)^2\right]\frac{\partial}{\partial z}, \\
\Gamma_8 &= a\frac{\partial}{\partial x} - u(u+v)a_\xi\frac{\partial}{\partial u} - v(u+v)a_\xi\frac{\partial}{\partial v}, \\
\Gamma_9 &= (u+v)b_\xi\frac{\partial}{\partial u} - (u+v)b_\xi\frac{\partial}{\partial v} + b\frac{\partial}{\partial w} - b\frac{\partial}{\partial z},
\end{aligned}$$

where $a = a(\xi)$, $b = b(\xi)$, $\xi = w + z$ satisfy $a_t + b_{\xi\xi} = 0$, $b_t + \mu a_{\xi\xi} = 0$. Here symmetries $\Gamma_1 - \Gamma_4$ project into point symmetries of (1), while $\Gamma_5 - \Gamma_9$ induce potential symmetries of (1).

3 Linearizing mappings

In this section we adopt the idea of Bluman et. al [3] who use infinite–dimensional potential symmetries to derive a nonlocal mapping that linearizes equation (10). We consider one special case of (1). Similarly, many other cases that admit infinite–dimensional potential symmetries can be linearized. Here we consider the nonlinear system

$$\begin{aligned} u_t &= \left[\frac{u}{v^2}u_x + \frac{\mu^2-u^2}{v^3}\,v_x\right]_x\,, \\ v_t &= \left[\frac{1}{v}\,u_x - \frac{u}{v^2}\,v_x\right]_x\,. \end{aligned} \tag{17}$$

We note that equation (17) is the example 5 in the previous section.

In section 2 we have seen that the auxiliary system

$$\begin{aligned} w_x &= u\,, \\ w_t &= \frac{u}{v^2}\,u_x + \frac{\mu^2-u^2}{v^3}\,v_x\,, \\ z_x &= v\,, \\ z_t &= \frac{1}{v}\,u_x - \frac{u}{v^2}\,v_x \end{aligned} \tag{18}$$

of (17) admits two infinite–dimensional symmetries, Γ_8 and Γ_9. These two symmetries lead to the mapping

$$x' = z\,, \quad t' = \mu t\,, \quad w' = w - \mu x\,, \quad z' = w + \mu x\,, \quad u' = \frac{u-\mu}{v}\,, \quad v' = \frac{u+\mu}{v} \tag{19}$$

that connects the linear system

$$\begin{aligned} w'_{x'} &= u'\,, \\ w'_{t'} &= u'_{x'}\,, \\ z'_{x'} &= v'\,, \\ z'_{t'} &= -v'_{x'} \end{aligned}$$

and the nonlinear system (18). The inverse of the mapping (19)

$$x = \frac{z'-w'}{2\mu}\,, \quad t = \frac{t'}{\mu}\,, \quad w = \frac{z'+w'}{2}\,, \quad z = x'\,, \quad u = \frac{\mu(u'+v')}{v'-u'}\,, \quad v = \frac{2\mu}{v'-u'}$$

leads to the contact transformation

$$\begin{aligned} \mathrm{d}x &= \frac{1}{2\mu}\left[(v'-u')\mathrm{d}x' - (v'_{x'}+u'_{x'})\mathrm{d}t'\right], \\ \mathrm{d}t &= \frac{1}{\mu}\,\mathrm{d}t'\,, \\ u &= \frac{\mu(u'+v')}{v'-u'}\,, \\ v &= \frac{2\mu}{v'-u'}\,, \end{aligned} \tag{20}$$

which connects the linear system

$$\begin{aligned} u'_{t'} &= u'_{x'x'}\,, \\ v'_{t'} &= -v'_{x'x'} \end{aligned} \tag{21}$$

and the nonlinear system (17). We note that the first equation of the system (21) is the linear heat equation, while the second is the backward heat equation. Such equations are of importance in mathematical biology, for example in processes of chemo–taxis.

Now consider the hodograph–type transformation

$$x' = z\,, \quad t' = \mu t\,, \quad w' = w - \mu x\,, \quad z' = w + \mu x\,,$$

which leads to the contact transformation

$$\begin{aligned} \mathrm{d}x' &= v\,\mathrm{d}x + \left(v^{-1}u_x - uv^{-2}v_x\right)\mathrm{d}t\,, \\ \mathrm{d}t' &= \mu\,\mathrm{d}t\,, \\ u' &= \int u\,\mathrm{d}x - \mu x\,, \\ v' &= \int u\,\mathrm{d}x + \mu x\,, \end{aligned} \tag{22}$$

which connects the linear system (21) and the nonlinear system (17).

The composition of the two non-local transformations (20) and (22) produces the generalized hodograph–type transformation

$$\begin{aligned} \mathrm{d}x &\mapsto xv\,\mathrm{d}x + \left(v^{-1}u_x - uv^{-2}v_x - uv^{-1}\right)\mathrm{d}t\,, \\ \mathrm{d}t &\mapsto \mathrm{d}t\,, \\ u &\mapsto \frac{1}{x}\int u\,\mathrm{d}x\,, \\ v &\mapsto \frac{1}{x} \end{aligned} \tag{23}$$

that leaves invariant the system (17).

The generalized hodograph–type transformation (23) can generate new solutions for the system (17) using known solutions. For example, consider the trivial solution

$$u = c_1\,, \quad v = c_2\,.$$

Transformation (23) leads to the new solution

$$u = \frac{c_1\sqrt{2}\sqrt{c_2x + c_1t} + c_2c_3}{\sqrt{2}\sqrt{c_2x + c_1t}}\,, \quad v = \frac{c_2}{\sqrt{2}\sqrt{c_2x + c_1t}}\,.$$

This new solution can then be used to generate a second new solution and so on.

4 Remarks

In this paper we have considered the problem of finding nonlocal symmetries, known as potential symmetries, for systems of the form (1). We have presented certain special cases of (1) that admit potential symmetries. In particular, these special systems admit infinite dimensional potential symmetries. Such symmetries can then been used to derive nonlocal mappings that linearize the system. The present work can be seen as an initial point for further study. We give below some examples of further research that can be carried out in the same spirit as the present paper.

For equation (4) a complete classification of potential symmetries is achieved [3]. The corresponding problem for the system (1) remains open. Furthermore an interesting and important task is to determine all special forms of (1) that can be linearized.

Potential symmetries of (10) that correspond to the auxiliary system (6) have been listed in the Introduction. We note the second equation of the system (6) can be written in a conserved form. Introducing the potential variable w we have the second generation auxiliary system of (10)

$$v_x = u\,, \quad w_x = v\,, \quad w_t = -\frac{1}{u}\,.$$

Symmetries of the above system induce the potential symmetry

$$\Gamma = (w - 2xv)\frac{\partial}{\partial x} + u(2xu + v)\frac{\partial}{\partial u} + 2t\frac{\partial}{\partial v} - (v^2 - 2t)\frac{\partial}{\partial w}$$

of equation (10). Similarly, we can search for potential symmetries of (1) that correspond to higher generation auxiliary systems. In section 2 we have determined potential symmetries with the employment of first and second generation systems, respectively. For certain forms of (1) one or two equations can be written in a conserved form. For example, consider the example 1 of subsection 2.2. That is, the system (3) takes the form

$$\begin{aligned}
w_x &= u\,,\\
w_t &= \frac{u}{v^2}\,u_x - \frac{u^2}{v^3}\,v_x = \frac{1}{2}\left[\frac{u^2}{v^2}\right]_x\,,\\
z_x &= v\,,\\
z_t &= \frac{1}{v}\,u_x - \frac{u}{v^2}\,v_x = \left[\frac{u}{v}\right]_x\,.
\end{aligned}$$

We note that the second and fourth equations are written in conserved form. We can therefore introduce potential variables ρ or/and σ so that the above system system can be written as a system of five or six equations. Hence we have the following two third generation auxiliary systems of (1)

$$w_x = u\,,$$

$$
\begin{aligned}
\rho_x &= w\,,\\
\rho_t &= \frac{1}{2}\,\frac{u^2}{v^2}\,,\\
z_x &= v\,,\\
z_t &= \frac{1}{v}\,u_x - \frac{u}{v^2}\,v_x
\end{aligned}
$$

and

$$
\begin{aligned}
w_x &= u\,,\\
w_t &= \frac{u}{v^2}\,u_x - \frac{u^2}{v^3}\,v_x\,,\\
z_x &= v\,,\\
\sigma_x &= z\,,\\
\sigma_t &= \frac{u}{v}
\end{aligned}
$$

and the fourth generation potential system

$$
\begin{aligned}
w_x &= u\,,\\
\rho_x &= w\,,\\
\rho_t &= \frac{1}{2}\,\frac{u^2}{v^2}\,,\\
z_x &= v\,,\\
\sigma_x &= z\,,\\
\sigma_t &= \frac{u}{v}\,.
\end{aligned}
$$

Now we can search for Lie symmetries for the above three systems with the ultimate goal that a number of these symmetries will induce potential symmetries for the original system (1).

Finally, we proposed the problem of finding hodograph–type transformations that linearize the potential system (3) and consequently the integrated (potential) form of the system (1)

$$
\begin{aligned}
w_t &= f(w_x, z_x)w_{xx} + g(w_x, z_x)z_{xx}\,,\\
z_t &= h(w_x, z_x)w_{xx} + k(w_x, z_x)z_{xx}\,.
\end{aligned}
$$

That is, to construct transformations similar to pure hodograph transformation that linearizes system (6) and consequently equation (12). We give two examples.

The transformation

$$
x' = z\,, \quad t' = t\,, \quad u' = \frac{1}{v}\,, \quad v' = \frac{u}{v}\,, \quad w' = x\,, \quad z' = w
$$

maps the linear system

$$
w'_{x'} = u'\,,
$$

$$\begin{aligned} w'_{t'} &= u'_{x'} + \mu_2 v'_{x'}\,, \\ z'_{x'} &= v'\,, \\ z'_{t'} &= \mu_3 u'_{x'} + \mu_4 v'_{x'} \end{aligned}$$

into the nonlinear system

$$\begin{aligned} w_x &= u\,, \\ w_t &= \frac{\mu_4 - \mu_2 u}{v^2}\, u_x + \frac{\mu_2 u^2 + (1-\mu_4)u - \mu_3}{v^3}\, v_x\,, \\ z_x &= v\,, \\ z_t &= -\frac{\mu_2}{v}\, u_x - \frac{\mu_2 u + 1}{v^2}\, v_x\,. \end{aligned}$$

Consequently, the hodograph–type transformation

$$x' = z\,, \quad t' = t\,, \quad w' = x\,, \quad z' = w$$

maps the linear system

$$\begin{aligned} w'_{t'} &= u'_{x'x'} + \mu_2 v'_{x'x'}\,, \\ z'_{t'} &= \mu_3 u'_{x'x'} + \mu_4 v'_{x'x'} \end{aligned}$$

into the nonlinear system

$$\begin{aligned} w_t &= \frac{\mu_4 - \mu_2 u_x}{v_x^2}\, u_{xx} + \frac{\mu_2 u_x^2 + (1-\mu_4)u_x - \mu_3}{v_x^3}\, v_{xx}\,, \\ z_t &= -\frac{\mu_2}{v_x}\, u_{xx} - \frac{\mu_2 u_x + 1}{v_x^2}\, v_{xx}\,. \end{aligned} \tag{24}$$

A second example is the hodograph–type transformation

$$x' = z\,, \quad t' = t\,, \quad w' = w\,, \quad z' = x$$

that maps the linear system

$$\begin{aligned} w'_{t'} &= \mu_4 u'_{x'x'} + \mu_3 v'_{x'x'}\,, \\ z'_{t'} &= \mu_2 u'_{x'x'} + v'_{x'x'} \end{aligned}$$

into the nonlinear system (24).

We note that the pure hodograph transformation is a cyclic group of order 2 (twice application of it gives the identity transformation). In the first example the hodograph–type transformation is a group of order 3 while in the second example is of order 2.

Another open problem is the search of approximate potential symmetries for the system (1). This can be done in the spirit of the work of Baikov et al. [13].

References

[1] J.R. Philip and D.A. De Vries: Trans. Am. Geophys. Un. **38** (1957) 222.

[2] W.A. Jury, J. Letey and L.H. Stolzy: Flow of water and energy under desert conditions in *Water in Desert Ecosystems*. (ed. D D Evans and J L Thames) Stroudsburg, PA: Dowden, Hutchinson and Ross, 1981, p. 92.

[3] G.W. Bluman and S. Kumei: *Symmetries and Differential Equations*. Springer-Verlag, New York, 1989.

[4] N.H. Ibragimov: *Elementary Lie Group Analysis and Ordinary Differential Equations*. Wiley, New York, 1999.

[5] R.J. Wiltshire: J. Phys. A: Math. Gen. **27** (1994) 7821.

[6] V.A. Baikov, A.V. Gladkov and R.J. Wiltshire: J. Phys. A **31** (1998) 7483.

[7] G.W. Bluman, G.J. Reid and S. Kumei: J. Math. Phys. **29** (1988) 806.

[8] G.W. Bluman: Potential symmetries and equivalent conservation laws, in *Modern group analysis: advanced analytical and computational methods in mathematical physics*. Acireale, 1992 (ed N.H. Ibragimov, M. Torrisi and A. Valenti), Kluwer Acad. Publ., Dordrecht, 1993, p. 71.

[9] C. Sophocleous: J. Phys A.: Math. Gen. **29** (1996) 6951.

[10] N.H. Ibragimov: *Lie Group Analysis of Differential Equations — Symmetries, exact solutions and conservation laws*. Vol. 1, Chemical Rubber Company, Boca Raton, FL, 1994.

[11] L.L. Ovsjannikov: Dokl. Akad. Nauk SSSR **125** (1959) 492 (in Russian).

[12] N. Euler, M.L. Gandarias, M. Euler and O. Lindblom: J. Math. Anal. Appl. **257** (2001) 21.

[13] V.A. Baikov, R.K. Gazizov and N.H. Ibragimov: Jnl. of Soviet Math. **55** (1991) 1450.