On quantum mechanics as a constrained deterministic dynamics

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In this paper we review recent results obtained in [quant-ph/0504200] on the path integral formulation of 't Hooft's derivation of quantum from classical physics. In particular, we employ the Faddeev–Jackiw treatment of classical constrained systems to show how 't Hooft's loss of information condition may yield a genuine quantum mechanical system. With two simple examples we discuss some of the consequences that follow from our approach.

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1 Introduction

Gerard 't Hooft, in a speculative 1988 paper [1], suggested that a suitably defined deterministic, local reversible cellular automata might provide a viable formalism for constructing quantum mechanics. In the paper he introduced several "toy models" that hinted that typically non-local, non-deterministic quantum mechanical behavior can be achieved using local deterministic laws, in a sense that a basis can be found in terms of which the wave function does not spread. Similar conclusions were also reported by other authors [2]. In his subsequent works [3] 't Hooft's substantially enlarged the class of emergent quantum dynamics by implementing an appropriate constraining procedure — loss of information condition, that accounted also for irreversible cellular automata.

Recently [4, 5], some progress has been made in formulating an alternative of the above proposal for continuous degrees of freedom. The key idea is that quantum mechanics represents merely the low-energy limit of some more fundamental continuous deterministic dynamics. In particular, by resorting to simple dynamical systems, it has been shown that a suitable constraining procedure applied to an appropriate continuous deterministic system, can reduce the physical degrees of freedom so that quantum mechanics emerges. Such a reduction of the degrees of freedom may be physically implemented by a mechanism of information loss or dissipation directly on the level of particle trajectories. This scheme has been further developed by several authors [4–10].

In the present paper we review the main results of our previous work [11], in which 't Hooft's quantization proposal was formulated in the language of path integrals. To deal with the singular nature of a dynamics inherent in such a proposal we use the Faddeev–Jackiw treatment [12] of constrained systems. A parallel approach based on the Dirac–Bergmann technique [13] can be found in Ref. [6]. The constrained dynamics enters into 't Hooft's scheme in two places: first, in the classical starting Hamiltonian which is of first order in the momenta and thus singular in the Dirac–Bergmann sense. Secondly, in the information loss condition that one has to enforce in order to achieve quantization [11]. Once the constraining condition and dynamics are specified, the Faddeev–Jackiw machinery can be introduced in order to identify the physical degrees of freedom. The path-integral formalism is then formulated on the reduced configuration space.

The rest of the paper goes as follows. In Section 2 we review some essentials of 't Hooft's quantization proposal for continuous degrees of freedom. To show these ideas more concrete we utilize in Section 3 a Lagrangian formulation of 't Hooft's Hamiltonian systems. With the help of the Faddeev–Jackiw procedure we quantize the latter through configuration-space path integrals. We show that the Faddeev–Jackiw mechanism is also useful in implementing the information-loss condition which is responsible for the emergent quantum behavior. In Section 4 we present two simple toy model examples elucidating our formalism. We conclude in Section 5.

2 't Hooft's proposal for continuous degrees of freedom

Let us start with a brief review of the main aspects of 't Hooft's proposal [4, 5]. The basic idea is that there exists a simple class of deterministic systems that can be described by means of Hilbert space techniques without loosing their deterministic character. Only after enforcing certain constraints expressing information loss, one obtains *bona fide* quantum systems. In such cases the quantum states of actually observed degrees of freedom (*observables*) can be identified with equivalence classes of states that span the original (primordial) Hilbert space of truly existing degrees of freedom (*be-ables*). It is important to understand that be-ables are not referring to conventional *macroscopic* variables, such as pointer on a detection device, but rather to a set of what 't Hooft calls "primordial" variables. Conventional variables, like mass, energy, position, etc., are viewed as emergent (non-primordial) degrees of freedom that mix different primordial states.

The above scenario has been successfully applied in discrete-time systems. For instance, in cellular automata with embedded information loss [1] the equivalence classes were invoked to obtain a unitary evolution operator which defines a genuine quantum mechanical Hamiltonian. Further examples of applications that involve 't Hooft proposal in discrete-time systems can be found, e.g. in Refs. [7, 10].

In the following we will consider only the case with continuous time. To this end we observe that classical systems of the form

$$H(\boldsymbol{p}, \boldsymbol{q}) = f^a(\boldsymbol{q}) p_a \qquad (1)$$

with repeated indices summed, evolve deterministically even after quantization [5]. This happens since in the Hamiltonian equations of motion

$$\dot{q}^a = \{q^a, H\} = f^a(\boldsymbol{q})\,, \tag{2}$$

$$\dot{p}_a = \{p_a, H\} = -p_b\, \frac{\partial f^b(\boldsymbol{q})}{\partial q^a}\,, \tag{3}$$

the equation for the q^a does not contain p_a, making the q^a *be-ables*. Because of the autonomous character of the dynamical equations (2) we can always decide to define a formal Hilbert space spanned by the states $\{|\boldsymbol{q}\rangle\}$, and define the associated momenta $\hat{p}_a = -\mathrm{i}\partial/\partial q^a$. The quantum mechanical "Hamiltonian" generating (2) is then $\hat{H} = f^a(\hat{\boldsymbol{q}})\hat{p}_a$. Indeed, due to linearity of $\hat{H}$ in $\hat{p}_a$ we have that $\hat{q}^a(t+\Delta t) = F^a[\hat{\boldsymbol{q}}(t), \Delta t]$ (F^a is some function) and hence $[\hat{q}^a(t), \hat{q}^b(t')] = 0$ for any t and t'. This in turn implies that the Heisenberg equation of the motion for $\hat{q}^a(t)$ in the $\boldsymbol{q}$-representation is identical with the c-number dynamical equation (2).

The basic physical problem with systems described by the Hamiltonian (1) is that they are not bounded from below. This defect can be repaired in the following way [5]: Let $\rho(\hat{\boldsymbol{q}})$ be some positive function of $\hat{q}_a$ with $[\hat{\rho}, \hat{H}] = 0$. Then we perform splitting

$$\begin{aligned} &\hat{H} = \hat{H}_+ - \hat{H}_-\,, \\ &\hat{H}_+ = \tfrac{1}{4}\,\hat{\rho}^{-1}\left(\hat{\rho} + \hat{H}\right)^2, \qquad \hat{H}_- = \tfrac{1}{4}\,\hat{\rho}^{-1}\left(\hat{\rho} - \hat{H}\right)^2, \end{aligned} \tag{4}$$

where $\hat{H}_+$ and $\hat{H}_-$ are positive definite operators satisfying

$$[\hat{H}_+, \hat{H}_-] = [\hat{\rho}, \hat{H}] = 0\,. \tag{5}$$

We may now employ the Dirac canonical quantization of constrained systems and enforce a lower bound upon the Hamiltonian by imposing the restriction

$$\hat{H}_-|\psi\rangle = 0 \tag{6}$$

on the Hilbert space of *be-ables*. The resulting *physical* state space, i.e. the space of *observables* has the energy eigenvalues that are trivially positive owing to

$$\hat{H}|\psi\rangle = \hat{H}_+|\psi\rangle = \hat{\rho}|\psi\rangle\,.$$

Concomitantly, in the Schrödinger picture the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi_t\rangle = -\mathrm{i}\hat{H}_+|\psi_t\rangle\,,$$

has only positive frequencies on physical states. Note that due to condition (5) 't Hooft's constraint (6) is a first-class constraint. From the theory of constrained systems [14] it is known that first-class conditions generate a gauge transformation and thus not only restrict the full Hilbert space but also produce equivalence classes of states. It should be noticed that above equivalence classes are generally non-local, in the sense that two states belong to the same class if they can be transformed into each other by the gauge transformation with the generator $\hat{H}_-$. If, in addition, the ensuing fiber-bundle structure is non-trivial one may encounter signatures of this through the emergence of geometric phases.

't Hooft proposed in Ref. [5] that in cases when the dynamical equations (2) describe the configuration-space chaotic dynamical system, the equivalent classes could be related to its stable orbits (e.g., limit cycles). The mechanism responsible for clustering of trajectories to equivalence classes was identified by 't Hooft as information loss — after while one cannot retrace back the initial conditions of a given trajectory, one can only say at what attractive trajectory it will end up. As the mechanism of equivalent classes is embodied in Eq.(6) we shall henceforth refer to it as *information loss condition*. Some applications of the the outlined scenario were given, e.g. in Refs. [10].

3 Path-integral formulation of 't Hooft's proposal

Because Feynman's path integrals [15] represent a legitimate alternative to canonical quantization, it is of some value to formulate 't Hooft's proposal in the language of path integrals. This will make available the powerful tools of the path-integral formalism and it will allow to incorporate the loss of information condition in a straightforward manner.

3.1 't Hooft's systems and path-integral quantization

We consider systems described by Hamiltonians of the type (1). Because of the absence of a leading kinetic term quadratic in the momenta p_a, the system classify as *singular* and the ensuing quantization can be done through some standard technique for quantization of constrained systems.

Particularly convenient is the technique proposed by Faddeev and Jackiw [12]. There one starts by observing that a Lagrangian for 't Hooft's equations of motion (2), (3) can be simply taken as

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{p}, \dot{\boldsymbol{p}}) = \boldsymbol{p} \cdot \dot{\boldsymbol{q}} - H(\boldsymbol{p}, \boldsymbol{q}) \tag{7}$$

with $\boldsymbol{q}$ and $\boldsymbol{p}$ being *Lagrangian variables* (in contrast to phase-space variables). Note that L does not depend on $\dot{\boldsymbol{p}}$. It is easily seen that the Euler–Lagrange equations for the Lagrangian (7) coincide with the Hamiltonian equations (2), (3). Thus given 't Hooft's Hamiltonian (1) one can always construct a first-order Lagrangian (7) whose configuration space coincides with the Hamiltonian phase space. By defining $2N$ configuration-space coordinates as

$$\begin{aligned} \xi^a &= p_a\,, \quad a = 1, \ldots, N\,, \\ \xi^a &= q^a\,, \quad a = N+1, \ldots, 2N\,, \end{aligned}$$

the Lagrangian (7) can be cast into the more expedient form, namely (summation convention is understood)

$$L(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) = \tfrac{1}{2}\, \xi^a \boldsymbol{\omega}_{ab} \dot{\xi}^b - H(\boldsymbol{\xi})\,. \tag{8}$$

Here $\boldsymbol{\omega}$ is the $2N \times 2N$ symplectic matrix

$$\boldsymbol{\omega}_{ab} = \begin{pmatrix} 0 & \boldsymbol{I} \\ -\boldsymbol{I} & 0 \end{pmatrix}_{ab} ,$$

which has an inverse $\boldsymbol{\omega}_{ab}^{-1} \equiv \boldsymbol{\omega}^{ab}$. The equations of the motion read

$$\dot{\xi}^a = \boldsymbol{\omega}^{ab} \, \frac{\partial H(\boldsymbol{\xi})}{\partial \xi^b} \, , \tag{9}$$

indicating that there are no constraints on $\boldsymbol{\xi}$. Consequentlty the Faddeev–Jackiw procedure makes the system unconstrained, so that the path integral quantization may proceed in the standard way. The time evolution amplitude is simply [15]

$$\langle \boldsymbol{\xi}_2, t_2 | \boldsymbol{\xi}_1, t_1 \rangle = \mathcal{N} \int_{\boldsymbol{\xi}(t_1)=\boldsymbol{\xi}_1}^{\boldsymbol{\xi}(t_2)=\boldsymbol{\xi}_2} \mathcal{D}\boldsymbol{\xi} \exp\left[\frac{\mathrm{i}}{\hbar} \int_{t_1}^{t_2} \mathrm{d}t \, L(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) \right] , \tag{10}$$

where $\mathcal{N}$ is the usual normalization factor, and the measure stands for

$$\mathcal{N} \int_{\boldsymbol{\xi}(t_1)=\boldsymbol{\xi}_1}^{\boldsymbol{\xi}(t_2)=\boldsymbol{\xi}_2} \mathcal{D}\boldsymbol{\xi} = \mathcal{N} \int_{\boldsymbol{q}(t_1)=\boldsymbol{q}_1}^{\boldsymbol{q}(t_2)=\boldsymbol{q}_2} \mathcal{D}\boldsymbol{q} \mathcal{D}\boldsymbol{p} \, . \tag{11}$$

Since the Lagrangian (7) is linear in $\boldsymbol{p}$, we may integrate these variables out and obtain

$$\langle \boldsymbol{q}_2, t_2 | \boldsymbol{q}_1, t_1 \rangle = \mathcal{N} \int_{\boldsymbol{q}(t_1)=\boldsymbol{q}_1}^{\boldsymbol{q}(t_2)=\boldsymbol{q}_2} \mathcal{D}\boldsymbol{q} \prod_a \delta \left[\dot{q}^a - f^a(\boldsymbol{q}) \right] , \tag{12}$$

where $\delta\left[\boldsymbol{f}\right] \equiv \prod_t \delta(\boldsymbol{f}(t))$ is the functional version of Dirac's δ-function. Hence the system described by the Hamiltonian (1) retains its deterministic character even after quantization. The paths are squeezed onto the classical trajectories determined by the differential equations $\dot{\boldsymbol{q}} = \boldsymbol{f}(\boldsymbol{q})$. The time evolution amplitude (12) contains a sum over only the classical trajectories — there are no quantum fluctuations driving the system away from the classical paths, which is precisely what should be expected from a deterministic dynamics.

The amplitude (12) can be brought into another form by utilizing the identity

$$\delta \left[\boldsymbol{f}(\boldsymbol{q}) - \dot{\boldsymbol{q}} \right] = \delta \left[\boldsymbol{q} - \boldsymbol{q}_{\mathrm{cl}} \right] (\det M)^{-1} ,$$

where M is a functional matrix formed by the second functional derivatives of the action $\mathcal{A}[\boldsymbol{\xi}] \equiv \int \mathrm{d}t \, L(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}})$:

$$M_{ab}(t, t') = \left. \frac{\delta^2 \mathcal{A}}{\delta \xi^a(t) \, \delta \xi^b(t')} \right|_{\boldsymbol{q}=\boldsymbol{q}_{\mathrm{cl}}} . \tag{13}$$

The Morse index theorem ensures that for sufficiently short time intervals $t_2 - t_1$ (before the system reaches its first focal point), the classical solution with the initial condition $\boldsymbol{q}(t_1) = \boldsymbol{q}_1$ is unique. In such a case Eq. (12) can be brought to the form

$$\langle \boldsymbol{q}_2, t_2 | \boldsymbol{q}_1, t_1 \rangle = \tilde{\mathcal{N}} \int_{\boldsymbol{q}(t_1)=\boldsymbol{q}_1}^{\boldsymbol{q}(t_2)=\boldsymbol{q}_2} \mathcal{D}\boldsymbol{q} \, \delta \left[\boldsymbol{q} - \boldsymbol{q}_{\mathrm{cl}} \right] \tag{14}$$

with $\tilde{\mathcal{N}} \equiv \mathcal{N}/(\det M)$. Remarkably, the Faddeev–Jackiw treatment bypasses completely the discussion of constraints, in contrast with the conventional Dirac–Bergmann method [13, 14] where $2N$ (spurious) second-class primary constraints must be introduced to deal with 't Hooft's system, as done in [6].

Finally we mention an interesting implication of the result (14). If we had started in Eq.(12) with an external current

$$\tilde{L}(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) = L(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) + \mathrm{i}\hbar \boldsymbol{J} \cdot \boldsymbol{q}\,,$$

integrated again over $\boldsymbol{p}$, and took the trace over $\boldsymbol{q}$, we would end up with a generating functional

$$\mathcal{Z}_{\mathrm{CM}}[\boldsymbol{J}] = \tilde{\mathcal{N}} \int \mathcal{D}\boldsymbol{q}\, \delta\left[\boldsymbol{q} - \boldsymbol{q}_{\mathrm{cl}}\right] \exp\left[\int_{t_1}^{t_2} \mathrm{d}t\, \boldsymbol{J} \cdot \boldsymbol{q}\right]. \tag{15}$$

This coincides with the path-integral formulation of classical mechanics (CM) postulated by Gozzi *et al.* in Refs. [16].

3.2 Inclusion of information loss

In the preceding section we have observed that the Hamiltonian (1) is not bounded from below. This is true for any function $f^a(\boldsymbol{q})$. Hence, no deterministic system with dynamical equations $\dot{q}^a = f^a(\boldsymbol{q})$ can describe a stable *quantum world*. To deal with this situation we now employ 't Hooft's procedure of Section 2. We assume that the system (1) has n conserved irreducible charges C^i, i.e.

$$\{C^i, H\} = 0\,, \quad i = 1, \ldots, n\,. \tag{16}$$

A lower bound is enforced upon H, by imposing the condition that $H_- = 0$ on the physically accessible part of a phase space.

The splitting of H into H_- and H_+ is conserved in time provided that $\{H_-, H\} = \{H_+, H\} = 0$, which is ensured if $\{H_+, H_-\} = 0$. Since the charges C^i in (16) form an irreducible set, the Hamiltonians H_+ and H_- must be functions of the charges and H itself. There is a certain amount of flexibility in finding H_- and H_+. For definiteness we take the following choice

$$H_+ = \frac{(H + a_i C^i)^2}{4 a_i C^i}\,, \quad H_- = \frac{(H - a_i C^i)^2}{4 a_i C^i}\,, \tag{17}$$

where $a_i(t)$ are $\boldsymbol{q}$ and $\boldsymbol{p}$ independent. The lower bound is reached by choosing $a_i(t) C^i$ to be non-negative. Following 't Hooft we should select a combination of C^i which is $\boldsymbol{p}$-independent [this condition may not necessarily be achievable for general $f^a(\boldsymbol{q})$].

In the Dirac–Bergmann quantization approach used in our previous paper [6], the information loss condition (6) was a first-class primary constraint. The corresponding gauge freedom then needed to be removed by a gauge condition. In the Faddeev–Jackiw approach, Dirac's elaborate classification of constraints to first or second class, primary or secondary is avoided. It is therefore worthwhile to rephrase the entire development in Ref. [6] in this new framework. The information loss condition may now be introduced by simply adding to the Lagrangian (8) a term enforcing

$$H_{-}(\boldsymbol{\xi}) = 0 \tag{18}$$

by means of a Lagrange multiplier. More in general we can take instead of H_{-} any function $\phi(\boldsymbol{\xi})$, such that $\phi(\boldsymbol{\xi}) = 0$ implies $H_{-}(\boldsymbol{\xi}) = 0$. In this way we obtain

$$L(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) = \tfrac{1}{2} x i^a \boldsymbol{\omega}_{ab} \dot{\xi}^b - H(\boldsymbol{\xi}) - \eta\, \phi(\boldsymbol{\xi})\,, \tag{19}$$

In Faddeev–Jackiw method one directly applies the constraint and thus eliminates one of ξ^a, say ξ^1, in terms of the remaining coordinates. This reduces the dynamical variables to $2N - 1$. Modulo an irrelevant total derivative, the canonical term $\xi^a \boldsymbol{\omega}_{ab} \dot{\xi}^b$ changes to $\xi^i \boldsymbol{f}_{ij}(\hat{\boldsymbol{\xi}}) \dot{\xi}^j$ with

$$\boldsymbol{f}_{ij}(\hat{\boldsymbol{\xi}}) = \boldsymbol{\omega}_{ij} - \left[\boldsymbol{\omega}_{1i} \frac{\partial \xi^1}{\partial \xi^j} - (i \leftrightarrow j) \right]. \tag{20}$$

Here $i, j = 2, \ldots, 2N$, and $\hat{\boldsymbol{\xi}} = \{\xi^2, \ldots, \xi^{2N}\}$. Eliminating ξ^1 also in the Hamiltonian H we obtain the reduced Hamiltonian $H_R(\hat{\boldsymbol{\xi}})$, so that we are left with the reduced Lagrangian

$$L_R(\hat{\boldsymbol{\xi}}, \dot{\hat{\boldsymbol{\xi}}}) = \tfrac{1}{2}\, \xi^i \boldsymbol{f}_{ij}(\hat{\boldsymbol{\xi}}) \dot{\xi}^j - H_R(\hat{\boldsymbol{\xi}})\,. \tag{21}$$

At this point one must worry about the notorious operator-ordering problem, not knowing in which temporal order $\hat{\boldsymbol{\xi}}$ and $\dot{\hat{\boldsymbol{\xi}}}$ must be taken in the kinetic term. A path integral in which the kinetic term is coordinate-dependent can in general only be defined perturbatively, and all anharmonic terms are then treated as interactions. $\mathcal{Z}_{\rm CM}$ is expanded in powers of expectation values of products of these interactions which, in turn, are expanded into integrals over all Wick contractions, the Feynman integrals. Each contraction represents a Green function. For the Lagrangian of the form (21), the contractions of two ξ^i's contain a Heaviside step function, those of one ξ^i and one $\dot{\xi}^i$ contain a Dirac δ-function, and those of two $\dot{\xi}^i$'s contain a function $\dot{\delta}(t - t')$. Thus, the position-space Feynman integrals run over products of distributions and are mathematically undefined. Fortunately, a unique definition has recently been put forward. It is enforced by the necessary physical requirement that path integrals must be invariant under coordinate transformations [17]. For the model systems discussed below (see Section 4) the ordering issue was extensively discussed in Ref. [11].

The Lagrangian is processed further with the help of Darboux's theorem [18]. This ensures an existence of a non-canonical (local) transformation $\xi^i \mapsto \{\zeta^s, z^r\}$ which brings L_R to the canonical form

$$L_R(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, \boldsymbol{z}) = \tfrac{1}{2}\, \zeta^s \boldsymbol{\omega}_{st} \dot{\zeta}^t - H_R'(\boldsymbol{\zeta}, \boldsymbol{z})\,, \tag{22}$$

where $\boldsymbol{\omega}_{st}$ is the canonical symplectic matrix in the reduced s-dimensional space. The variables z^r are related to zero modes of the matrix $\boldsymbol{f}_{ij}(\hat{\boldsymbol{\xi}})$ which makes it non-invertible. Each zero mode corresponds to a constraint of the system. In Dirac's language these would correspond to the secondary constraints. Since there is no $\dot{z}^r$ in the Lagrangian, the variables z^r do not play any dynamical rôle and can be eliminated using the equations of motion

$$\frac{\partial H'_R(\boldsymbol{\zeta},\boldsymbol{z})}{\partial z^r} = 0\,. \tag{23}$$

In general, $H'_R(\boldsymbol{\zeta},\boldsymbol{z})$ is a nonlinear function of z^{r_1}. One now solves as many z^{r_1} as possible in terms of remaining z's, which we label by z^{r_2}, i.e.

$$z^{r_1} = \varphi^{r_1}(\boldsymbol{\zeta}, z^{r_2})\,. \tag{24}$$

If $H'_R(\boldsymbol{\zeta},\boldsymbol{z})$ happens to be linear in z^{r_2}, we obtain the constraints

$$\varphi_{r_2}(\boldsymbol{\zeta}) = 0\,. \tag{25}$$

Substituting (24) into (22) we arrive at

$$L_R(\boldsymbol{\zeta},\dot{\boldsymbol{\zeta}},\boldsymbol{z}) = \tfrac{1}{2}\,\zeta^s\boldsymbol{\omega}_{st}\dot{\zeta}^t - H''_R(\boldsymbol{\zeta}) - z^{r_2}\varphi_{r_2}(\boldsymbol{\zeta})$$

with z^{r_2} playing the rôle of Lagrange multipliers. We now repeat the elimination procedure until there are no more z-variables. The surviving variables represent the true physical degrees of freedom. In the Dirac–Bergmann approach, these would span the *reduced* phase space Γ^*.

Let us follow the procedure in more detail if there is just one variable z in (23) and only equation (24) holds. As in Ref. [6], we can pass to the new set of canonical variables $\boldsymbol{\xi} \mapsto \{\boldsymbol{\zeta}, z, p_z\}$ with $p_z = \phi$. Let us define the function

$$\chi(\boldsymbol{\zeta},z) \equiv \frac{\partial H'_R(\boldsymbol{\zeta},z)}{\partial z} = \frac{\partial H_+(\xi^1(\hat{\boldsymbol{\xi}}),\hat{\boldsymbol{\xi}})}{\partial z} = \{H_+,\phi\}\Big|_{p_z=0} = 0\,. \tag{26}$$

Its derivative is given by the Poisson bracket

$$\frac{\partial \chi(\boldsymbol{\zeta},z)}{\partial z} = \{\chi(\boldsymbol{\zeta},z),p_z\} = \{\chi,\phi\} \neq 0\,. \tag{27}$$

Because (27) is different from zero on account of (24) we can identify the function $\chi(\boldsymbol{\zeta},z)$ with the implicit gauge fixing condition of the Faddeev–Jackiw analysis.

Let us now see how we can include the constraints (18) and (26) into the path integral (15) for $\mathcal{Z}_{\rm CM}[\boldsymbol{J}]$. This cannot simply be done by inserting δ-functionals $\delta[\phi]$ and $\delta[\chi]$ into the integrand, since ϕ and χ may not be independent. Allowing for this, the path integral reads (see Ref. [6])

$$\mathcal{Z}_{\rm CM}[\boldsymbol{J}] = \int \mathcal{D}\boldsymbol{\xi}\,\delta[\phi]\delta[\chi]\Big|\det\|\{\phi,\chi\}\|\Big|\exp\left[{\rm i}\int_{t_i}^{t_f}{\rm d}t\,L(\boldsymbol{\xi},\dot{\boldsymbol{\xi}}) + \int_{t_i}^{t_f}{\rm d}t\,\boldsymbol{J}\boldsymbol{\xi}\right]. \tag{28}$$

Assuming that ξ^1 can be eliminated globally from (19), we obtain

$$\mathcal{Z}_{\rm CM}[\boldsymbol{J}] = \int \mathcal{D}\hat{\boldsymbol{\xi}}\,\delta[\chi]\,\Big|\det\|\{\phi,\chi\}\|\Big|\left|\det\left\|\frac{\delta\phi}{\delta\xi^1}\right\|\right|^{-1}_{\xi^1=\xi^1(\hat{\boldsymbol{\xi}})}\times$$
$$\times\exp\left[{\rm i}\int_{t_i}^{t_f}{\rm d}t\,L_R(\hat{\boldsymbol{\xi}},\dot{\hat{\boldsymbol{\xi}}}) + \int_{t_i}^{t_f}{\rm d}t\,\boldsymbol{J}\boldsymbol{g}(\hat{\boldsymbol{\xi}})\right].$$

After the Darboux transformation, this becomes

$$\mathcal{Z}_{\rm CM}[\boldsymbol{J}] = \int \mathcal{D}\boldsymbol{\zeta}\, \mathcal{D}z\, \delta\left[z-\varphi(\boldsymbol{\zeta})\right] \exp\left[\mathrm{i}\int_{t_i}^{t_f} \mathrm{d}t\, L_R(\boldsymbol{\zeta},\dot{\boldsymbol{\zeta}},z) + \int_{t_i}^{t_f} \mathrm{d}t\, \boldsymbol{J}\boldsymbol{g}(\boldsymbol{\zeta},z)\right].$$

Here the functional relation

$$\delta[\chi]\left|\det\|\{\phi,\chi\}\|\right| = \delta[z-\varphi(\boldsymbol{\zeta})]$$

together with Jacobi–Liouville equality

$$\frac{\partial(\xi^2,\ldots,\xi^{2N})}{\partial(\zeta^1,\ldots,\zeta^{2N-2},z)} = \frac{\partial(\xi^2,\ldots,\xi^{2N},p_z)}{\partial(\zeta^1,\ldots,\zeta^{2N-2},z,p_z)}\,\frac{\partial(\zeta^1,\ldots,\zeta^{2N-2},z,p_z)}{\partial(\xi^2,\ldots,\xi^{2N},\xi^1)} = $$
$$= \left(\frac{\partial p_z}{\partial \xi^1}\right)_{\hat{\boldsymbol{\xi}}} = \left(\frac{\partial \phi}{\partial \xi^1}\right)_{\xi^1=\xi^1(\hat{\boldsymbol{\xi}})}$$

were used. With the notation $H_+^*(\boldsymbol{\zeta}) = H_+(\boldsymbol{\zeta}, z=\varphi(\boldsymbol{\zeta}), p_z=0)$, this can be rewritten as

$$\mathcal{Z}_{\rm CM}[\boldsymbol{J}] = \int \mathcal{D}\boldsymbol{\zeta} \exp\left[\mathrm{i}\int_{t_i}^{t_f} \mathrm{d}t\, \zeta^t \boldsymbol{\omega}_{ts}\dot{\zeta}^s\right] \exp\left[-\mathrm{i}\int_{t_i}^{t_f} \mathrm{d}t\, H_+^*(\boldsymbol{\zeta}) + \int_{t_i}^{t_f} \mathrm{d}t\, \boldsymbol{J}\boldsymbol{g}^*(\boldsymbol{\zeta})\right]. \tag{29}$$

At this point we note that the result (29) is equivalent to the result derived in Ref. [6]. In fact, when χ in [6] coincides with the the form (26) and we set $\boldsymbol{\zeta} = (\bar{\boldsymbol{Q}}, \bar{\boldsymbol{P}})$, $z = Q_1$, and $p_z = P_1$, then $\mathcal{Z}_{\rm CM}[\boldsymbol{J}]$ from Ref. [6] reduces exactly to the form (29).

Important simplification happens when H_R' is independent of z (e.g. when $\phi = H_-$). In Dirac–Bergmann's language this imply that there is no secondary constraint. In such a case the gauge fixing can be enforced by taking $\chi = z$ (see Ref. [19]), and the procedure outlined in steps (28)–(29) is streamlined by the fact that $\left|\det\|\{\phi,\chi\}\|\right| = 1$. The corresponding coordinate basis $\{\boldsymbol{\zeta},\chi,\phi\}$ is known as the Shouten–Eisenhart basis [14].

4 Examples of emergent quantum systems

4.1 Free particle

We conclude our presentation by exhibiting how our mathematical scheme works for a simple system described by 't Hooft's Hamiltonian

$$H(\boldsymbol{q},\boldsymbol{p}) = xp_y - yp_x\,. \tag{30}$$

Hamiltonian (30) formally represents the z component of the angular momentum, whose spectrum is clearly unbounded from below. One can also regard (30) as describing the mathematical pendulum. This is because the corresponding dynamical equation (2) for the $\boldsymbol{q}$-variable is a plane pendulum equation with the pendulum constant $l/g = 1$. The Lagrangian (7) then reads

$$L(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{p},\dot{\boldsymbol{p}}) = p_x\dot{x} + p_y\dot{y} - xp_y + yp_x\,.$$

It is well-known [6] that the system (30) has two independent constants of motion:

$$C_1 = x^2 + y^2\,, \quad C_2 = xp_x + yp_y\,.$$

However, only C_1 is $\boldsymbol{p}$-independent, so that the constraint $\rho(\boldsymbol{q})$ acquires the form: $\rho(\boldsymbol{q}) = a_1 C_1(\boldsymbol{q})$, with an arbitrary constant a_1. The reduced Lagrangian (22) reads

$$\begin{aligned} L_R(\hat{\boldsymbol{\xi}}, \dot{\hat{\boldsymbol{\xi}}}) &= \dot{y}p_y + \frac{\dot{x}}{y}\left(p_y x - a_1(x^2+y^2)\right) - a_1(x^2+y^2) = \\ &= \sqrt{x^2+y^2}\,\frac{\mathrm{d}}{\mathrm{d}t}\left[-2a_1\sqrt{x^2+y^2}\ \mathrm{arctg}\left(\frac{x}{y}\right) - \frac{xp_x + yp_y}{\sqrt{x^2+y^2}}\right] - a_1(x^2+y^2)\,. \end{aligned} \tag{31}$$

We can diagonalize the symplectic structure via the Darboux transformation

$$\begin{aligned} p_\zeta &= \sqrt{x^2+y^2}\,, \\ \zeta &= -2a_1\sqrt{x^2+y^2}\,\mathrm{arctg}\left(\frac{x}{y}\right) - \frac{xp_x + yp_y}{\sqrt{x^2+y^2}}\,. \end{aligned} \tag{32}$$

Hence, up to a total derivative, the reduced Lagrangian (31) goes over into

$$L_R(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, z) = \tfrac{1}{2}\,\zeta^s \boldsymbol{\omega}_{st}\dot{\zeta}^t - a_1(p_\zeta)^2 \tag{33}$$

with the symplectic notation $\zeta \equiv \zeta^1$ and $p_\zeta \equiv \zeta^2$. The reduced Hamiltonian is z-independent and thus $\chi = z$. Note that (32) together with

$$z = -\,\mathrm{arctg}\left(\frac{x}{y}\right) \quad \text{and} \quad p_z^2 = \phi^2 = 4a_1 p_\zeta^2\, H_-\,, \tag{34}$$

constitute the canonical transformation $\boldsymbol{\xi} \mapsto \{\boldsymbol{\zeta}, z, p_z\}$.

Due to a non-linear nature of the canonical transformation (32) and (34) one must check the path integral measure for a potential anomaly. In Ref. [11] it was shown that although the anomaly is indeed generated, it gets cancelled due to the presence of the constraining δ-functionals in the measure.

Let us now set $a_1 = \frac{1}{2}\, m\hbar$. After rescaling in the path integral the variable $\zeta(t)$ to $\zeta(t)/\hbar$ we obtain the correct path-integral measure of quantum systems:

$$\mathcal{D}\boldsymbol{\zeta} \approx \prod_i \left[\frac{\mathrm{d}\zeta(t_i)\,\mathrm{d}p_\zeta(t_i)}{2\pi\hbar}\right].$$

In addition, the prefactor $1/\hbar$ in the exponent emerges correctly. Thus, the classical partition function of Gozzi *et al.* morphs into the quantum partition function for a free particle of mass m. As the constant a_1 represents the choice of units for C_1 we see that the quantum scale $\hbar$ is implemented into the partition function via the information loss condition.

A free particle can emerge also from another class of 't Hooft's systems. Such systems can be obtained by modifying slightly the previous discussion and considering instead the Hamiltonian

$$H = xp_y - yp_x + \lambda(x^2+y^2)\,,$$

where λ is a constant. 't Hooft's information loss condition and $\rho(\boldsymbol{q})$ remain clearly the same as in the previous case. The reduced Lagrangian now reads

$$L_R(\hat{\boldsymbol{\xi}},\dot{\hat{\boldsymbol{\xi}}}) = \sqrt{x^2+y^2}\,\frac{\mathrm{d}}{\mathrm{d}t}\left[-2a_1\sqrt{x^2+y^2}\ \mathrm{arctg}\left(\frac{x}{y}\right) - \frac{xp_x+yp_y}{\sqrt{x^2+y^2}}\right] - a_1^*(x^2+y^2)$$

with $a_1^* = a_1 + \lambda$. Identical reasonings as in the preceding situation lead again to a proper quantum-mechanical partition function for a free particle.

4.2 Harmonic oscillator

In Ref. [6] it was shown that the system (30) can be also used to obtain the quantized linear harmonic oscillator. This is because there is a certain ambiguity in imposing 't Hooft's condition. This will be illustrated with $\phi = xp_y - yp_x - a_1(x^2+y^2)$ used in Eq. (34). The constraint $\phi = 0$ can be equivalently written as

$$\phi = \boldsymbol{x} \wedge \boldsymbol{A} = 0$$

with $\boldsymbol{x} = (x, y)$ and $\boldsymbol{A} = (p_x + a_1 y, p_y - a_1 x)$. The solution of $\phi = 0$ is formally given by

$$x = \alpha\,(p_x + a_1 y)\,, \quad y = \alpha\,(p_y - a_1 x)\,,$$

where α is an arbitrary real number. Note that $\alpha = 0$ and $\alpha = \infty$ also cover the singular cases $|\boldsymbol{x}| = 0$ and $|\boldsymbol{A}| = 0$, respectively. So, instead of one first-class condition $\phi = 0$ we can consider two second-class constraints

$$\begin{aligned}\phi_1 &= \left(p_x - \frac{x}{\alpha} + a_1 y\right) = 0\,,\\ \phi_2 &= \left(p_y - \frac{y}{\alpha} - a_1 x\right) = 0\end{aligned}$$

($\{\phi_1, \phi_2\} = 2a_1 \neq 0$). Equivalently one may view ϕ_1 as a primary first-class constraint and ϕ_2 as the gauge fixing condition. To make contact with the Faddeev–Jackiw procedure we chose the second scenario. The corresponding reduced Lagrangian is then

$$\begin{aligned}L_R(\hat{\boldsymbol{\xi}},\dot{\hat{\boldsymbol{\xi}}}) &= \dot{y}p_y + \dot{x}\left(\frac{x}{\alpha} - a_1 y\right) - xp_y + y\left(\frac{x}{\alpha} - a_1 y\right) = \\ &= -\frac{1}{2a_1}\left(p_y + a_1 x - \frac{y}{\alpha}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(p_x + a_1 y - \frac{x}{\alpha}\right) - xp_y + y\left(\frac{x}{\alpha} - a_1 y\right).\end{aligned} \tag{35}$$

At this point we can perform Darboux's transformation

$$\begin{aligned}p_\zeta &= \frac{1}{\sqrt{2}}\left(p_y + a_1 x - \frac{y}{\alpha}\right),\\ \zeta &= -\frac{1}{\sqrt{2}a_1}\left(p_x - \frac{x}{\alpha} - a_1 y\right),\\ z &= \frac{\phi_2}{2a_1} = \frac{1}{2a_2}\left(p_y - \frac{y}{\alpha} - a_1 x\right).\end{aligned} \tag{36}$$

The reduced Lagrangian (35) then becomes

$$L_R(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, z) = \tfrac{1}{2}\, \zeta^s \boldsymbol{\omega}_{st} \dot{\zeta}^t - \frac{1}{2a_1}\, p_\zeta^2 - \frac{a_1}{2}\left(\zeta^2 - 2z^2\right) \tag{37}$$

($\zeta \equiv \zeta^1$, $p_\zeta \equiv \zeta^2$). The stabilization condition $\chi(\boldsymbol{\zeta}, z) = 0$ yields then the gauge fixing

$$\chi(\boldsymbol{\zeta}, z) = \frac{\partial H'_R(\boldsymbol{\zeta}, z)}{\partial z} = -2a_1 z = 0\,.$$

By by enforcing a gauge constraint $z = 0$ in Eq.(37) we eliminate the variable z and obtain a non-degenerate reduced Lagrangian

$$L_R(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}) = \frac{1}{2}\, \zeta^s \boldsymbol{\omega}_{st} \dot{\zeta}^t - \frac{1}{2a_1}\, p_\zeta^2 - \frac{a_1}{2}\, \zeta^2\,.$$

The canonical transformation $\boldsymbol{\xi} \mapsto (\boldsymbol{\zeta}, z, p_z)$ is completed by identifying

$$p_z = -\phi_1 = -p_x - a_1 y + \frac{x}{\alpha}\,. \tag{38}$$

Similarly as in the previous case, $\{p_\zeta, \zeta, z, p_z\}$ can be identified with the Shouten–Eisenhart basis.

By choosing $a_1 = 1/m\hbar$ and rescaling $\zeta(t) \mapsto \zeta(t)/\hbar$ in the path integral (29) we obtain the quantum partition function for the linear harmonic oscillator with a unit frequency. One can again observe that the fundamental scale (suggestively denoted as $\hbar$) enters the partition function in a correct quantum mechanical manner. This is precisely the result which 't Hooft conjectured for the system (30) in Ref. [5]. Because the canonical transformation $\boldsymbol{\xi} \mapsto (\boldsymbol{\zeta}, z, p_z)$ is linear it does not induce anomaly in the path integral measure nor in the action (for discussion see Ref. [11]).

In the framework of the Dirac–Bergmann treatment both results discussed above were already derived in Ref. [6]. It is clear that other emergent quantum systems can be generated in an analogous manner. For instance, in Ref. [6] free particle weakly coupled to Duffing's oscillator was obtained from the Rössler system.

5 Conclusions

We have reviewed the main aspects of the path-integral formulation of 't Hooft's quantization proposal. We pointed out that the Faddeev–Jackiw treatment of constrained systems as discussed in Ref. [11] offers a series of advantages with respect to the usual Dirac–Bergmann scheme. Although both approaches require in 't Hooft's scheme a doubling of configuration-space degrees of freedom, it is the Faddeev–Jackiw scheme that does not explicitly invokes the classification of constraints into first and second class, primary and secondary [6], and that markedly simplifies the formal construction of path integrals for 't Hooft's constrained dynamics.

Our treatment of 't Hooft's quantization procedure is kept as simple as possible. For this purpose we advocate the Faddeev–Jackiw method as a convenient tool for imposing 't Hooft's information loss condition. The Faddeev–Jackiw treatment is a typical constrain-before-quantization method, and as such it is, at least formally, simpler than the Dirac–Bergmann method when path integrals are involved. This is because the Darboux transformation — the key element in the Faddeev–Jackiw method — formalizes structure of canonical transformations. On a practical level, however, the actual calculations seem to be more complicated than in Dirac–Bergmann way, as we could see in our simple examples. In particular, the change of coordinates (Darboux coordinates) from the pre-symplectic to a symplectic form plus non-dynamical $\boldsymbol{z}$-variables are often involved, or even impossible. Clearly, many alternatives to the Faddeev–Jackiw approach can be considered. One of them would be the field-antifield formalism of Batalin and Vilkovisky, which is presently under investigation.

The equivalence between both constraining approaches was discussed in Ref. [11]. There we have shown that under rather general conditions 't Hooft's quantization program performed with the Dirac–Bergmann and the Faddeev–Jackiw prescriptions leads to equivalent path integral representations of emergent quantum systems. Care is, however, needed when non-linear canonical transformations are performed in path integrals. Then the ordering issue and Gribov ambiguities start to be important and the above procedure necessarily involves non-trivial subtleties connected with the path-integral measure [11].

The present work is only a first step in describing more complicate primordial systems and ensuing emergent quantum dynamics. The logically next step would be a study of chaotic dynamical systems with non-trivial limit cycles. One can expect that dynamical systems satisfying the Poincaré-Bendixson theorem [20] could be good candidates for this task.

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References

[1] G. 't Hooft: J. Stat. Phys. **53** (1988) 323.

[2] E. Fredkin and T. Toffoli: Int. J. Theor. Phys. **21** (1982) 219;
E. Fredkin, Physica D (1990) 254;
K. Svozil: J. Phys. A: Math. Gen. **20** (1987) 3861.

[3] G. 't Hooft: Class. Quant. Grav. **13** (1996) 1023; Class. Quant. Grav. **16** (1999) 3263.

[4] G. 't Hooft: Int. J. Theor. Phys. **42** (2003) 355; [`hep-th/0105105`].

[5] G. 't Hooft: in *37th International School of Subnuclear Physics*, Erice, Ed. A. Zichichi, World Scientific, London, 1999.

[6] M. Blasone, P. Jizba and H. Kleinert: Phys. Rev. A **71** (2005) 052507.

[7] M. Blasone, P. Jizba and G. Vitiello: J. Phys. Soc. Jap. Suppl. **72** (2003) 50.

[8] H.T. Elze: Phys. Lett. A **310** (2003) 110; Physica A **344** (2004) 478; [hep-th/0411176]; [gr-qc/0307014]; Braz. J. Phys. **35** (2005) 343; Phys. Lett. A **335** (2005) 258; [hep-th/0508095].

[9] J.J. Halliwell: Phys. Rev. D **63** (2001) 085013.

[10] M. Blasone, P. Jizba and G. Vitiello: Phys. Lett. A **287** (2001) 205;
M. Blasone and P. Jizba: Can. J. Phys. **80** (2002) 645;
M. Blasone, E. Celeghini, P. Jizba and G. Vitiello: Phys. Lett. A **310** (2003) 393.

[11] M. Blasone, P. Jizba and H. Kleinert: [quant-ph/0504200]; accepted to Ann. of Phys.

[12] L.D. Faddeev and R. Jackiw: Phys. Rev. Lett. **60** (1988) 1692;
R. Jackiw: *Diverse Topics in Theoretical and Mathematical Physics*. World Scientific, Singapore, 1995.

[13] P.A.M. Dirac: Can. J. Math. **2** (1950) 129; Proc. R. Soc. London. A **246** (1958) 326; *Lectures in Quantum Mechanics*. Yeshiva University, New York, 1965;
P.G. Bergmann: Phys. Rev. **75** (1949) 680;
J.L. Anderson and P.G. Bergmann: Phys. Rev. **83** (1951) 1018.

[14] K. Sundermeyer: *Constrained Dynamics with Applications to Yang-Mills theory, General Relativity, Classical Spin, Dual String Model*. Springer Verlag, Berlin, 1982.

[15] H. Kleinert: *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*. World Scientific, Singapore 2004, Fourth extended edition, (http://www.physik.fu-berlin.de/˜kleinert/b5).

[16] E. Gozzi: Phys. Lett. B **201** (1988) 525;
E. Gozzi, M. Reuter and W.D. Thacker: Phys. Rev. D **40** (1989) 3363;
E. Gozzi and D. Mauro: [quant-ph/0507147];
E. Gozzi and D. Mauro: [quant-ph/0508199].

[17] H. Kleinert and A. Chervyakov: Phys. Lett. B **464**, 257 (1999); Phys. Lett. B **477** (2000) 373; Phys. Lett. A **273** (2000) 1; see also Chapter 10 of the textbook [15].

[18] A. Cannas da Silva: *Lectures on symplectic geometry*. Springer Verlag, Berlin, Heidelberg, 2001;
V. Guillemin and S. Sternberg: *Symplectic techniques in physics*. Cambridge University Press, New York, 1993.

[19] L.D. Faddeev: *Quantum Field Theory Lecture Notes*. (Lecture 4), (http://www.math.ias.edu/QFT/fall/index.html).

[20] see e.g. S.H. Strogatz: *Nonlinear dynamics and chaos*. Perseus, London, 1987.