# su(2|1) path integral for strongly correlated electrons: application to the pseudogap phenomenon

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We employ the su(2|1) superalgebra representation of the $t$–$J$ model Hamiltonian to rigorously enforce the local constraint that guarantees a given lattice site to be either empty or singly occupied. This constraint arises because a Coulomb repulsive energy dominates over hopping energy and results in strong electron correlation which determines the basic physics of high-$T_c$ superconductors. We apply this technique to derive a boson–spinless fermion model for the $t$–$J$ Hamiltonian, which provides a microscopic scenario to take into account local spin fluctuations to the pseudogap phenomenon.

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## 1 Introduction

It is now widely believed that the physics of high-$T_c$ cuprate superconductors on a 2D space lattice is basically governed by strong electron–electron correlations, which arise due to a strong Coulomb repulsion. Due to that repulsion two electrons cannot be simultaneously locate on one and the same lattice site, which completely blocks charge propagation even in a half-filled band, where each site is occupied by just one electron. This state known as a Mott insulator exhibits however strong short-range spin-spin electron correlations. Upon doping that ground state develops low-lying excitations, the magnetic correlations turn the system into a spin liquid and finally, charge and spin are recombined into quasiparticles leading to a new type of superconductivity.

To make any quantitative predictions within this picture, one has to take into account analytically the crucial constraint of no double occupancy (NDO). This will allow us to deal properly with the strong electron-electron correlations which are essential to describe the ground state of a Mott insulator. The mean field (MF) slave-boson/fermion theory is a commonly used approach to address the $t$–$J$ model when dealing with spin–charge separation in the context of a spin liquid, or the resonating valence bond (RVB) state. Within this scheme the electron operator is represented by a product of two commuting operators, that carry separately spin and charge degrees of freedom. Within this approach the NDO constraint transforms itself into the requirement that exactly only one field excitation can exist at each lattice site. Rigorous implementation of this local constraint of one particle per site poses however severe technical problem, whereas its MF treatment results in qualitatively different phase diagrams for slave fermions and slave bosons. Therefore it would be desirable to start with a theory where the local NDO constraint is taken into account rigorously prior to any MF approximations.

In the present paper we attempt to formulate such a theory. Technically we work within the $t$–$J$ model for the Hubbard operators, which enforces the local NDO constraint from the very beginning. Since the Hubbard operators are the generators of the su(2|1) superalgebra [1, 2], we employ the su(2|1) coherent-state path integral for the $t$–$J$ partition function. The effective action is then a function of the local coordinates of the SU(2|1) homogeneous coherent-state manifold which represents the phase space of the Hubbard operators.

To illustrate an application of this technique we show that within the RVB and linear spin wave (LSW) approximations this effective action reduces to that of the boson–spinless fermion model (BSFM). Spinless fermions emerge due to the NDO constraint. Unlike the standard phenomenological boson-fermion model (BFM), which is believed to capture the essential physics of the anomalous normal state of the underdoped cuprates, the boson–fermion interaction in the BSFM is a nonlocal function of the lattice sites due to the spinless character of the fermions. Besides, the origin of elementary excitations in BSFM is quite different from that in BFM.

Numerical calculations carried out in 1D case show however that BSFM has similar behavior to the BFM with respect to the pseudogap phenomenon which is a characteristic feature of the anomalous normal state of underdoped cuprates. Finally we emphasize that in contrast with the BFM, the BSFM is directly related to the $t$–$J$ model and has in turn a stronger microscopic basis.

Applications of our theory to describe the superconducting state of high-$T_c$ materials will be given elsewhere [3].

## 2 $t-J$ Hamiltonian and the NDO constraint

We start from the $t$–$J$ Hamiltonian on a square lattice [4]

$$H_{t-J} = P\biggl(-t\sum_{ij\sigma} c^{\dagger}_{i\sigma}c_{j\sigma} + \text{h.c.} + J\sum_{ij}\Bigl(\vec{Q}_i\vec{Q}_j - \tfrac{1}{4}\,n_i n_j\Bigr)\biggr)P \tag{1}$$

projected onto the space with no doubly occupied sites. Here $c_{i\sigma}$ is the electron annihilation operator at site $i$ with the spin projection $\sigma = \uparrow\downarrow$, $n_{i\sigma} = c^{+}_{i\sigma}c_{i\sigma}$, and $\vec{Q_i} = \frac{1}{2}\sum_{\sigma\sigma'} c^{\dagger}_{i\sigma}\vec{\tau}_{\sigma\sigma'}c_{i\sigma'}$ is the electron spin operator with the $\vec{\tau}$'s being Pauli matrices. Hamiltonian (1) contains a kinetic term of strength $t$ responsible for the hopping of electrons from one lattice site to its nearest neighbor, and a potential term of strength $J$ which describes nearest-neighbor spin exchange interaction. If $J \ll t$, the $t$–$J$ model can be mapped onto a Hubbard model with strong Coulomb repulsion. For other values of coupling constants those two models exhibit quite different behaviors.

At every lattice site the Gutzwiller projection operator $P = \prod_i (1 - n_{i\sigma}n_{i-\sigma})$ eliminates the doubly occupied states $|\uparrow\downarrow\rangle$ thereby reducing the quantum Hilbert space to a lattice site product of the 3-dimensional spaces spanned by the vectors $|0\rangle_i$, $|\uparrow\rangle_i$, $|\downarrow\rangle_i$. Physically this modification of the original Hilbert space takes into account the extra strong electron correlation effects in addition to the simple Pauli exclusion principle. It is this new type of correlation that is believed to account for the unusual and extremely rich physics of high-$T_c$ superconductors.

Explicitly, the projected electron operators take the form

$$Pc_{i\sigma}^{\dagger}P = c_{i\sigma}^{+}\,(1-n_{i-\sigma}) =: X_i^{\sigma 0}\,,$$

and are known as the Hubbard [5] or $X$-operators. Note that Hubbard operators are represented (at each lattice site) by 3×3 matrices. Note also that on the $P$-projected Hilbert space one has $n_{i\sigma}n_{i-\sigma}=0$.

The NDO constraint, $\sum_\sigma n_{i\sigma}\leq 1$, now holds rigorously at each lattice site and in terms of the Hubbard operators, the $t$–$J$ model becomes

$$H_{t-J} = -t\sum_{ij\sigma} X_i^{\sigma 0}X_j^{0\sigma} + \text{h.c.} + J\sum_{ij}\left(\overrightarrow{Q}_i\overrightarrow{Q}_j - \tfrac{1}{4}\,n_in_j\right), \tag{2}$$

where the electron spin operator now reads $\overrightarrow{Q_i} = \frac{1}{2}\sum_{\sigma\sigma'} X_i^{\sigma 0}\overrightarrow{\tau}_{\sigma\sigma'}X_i^{0\sigma'}$.

The fermionic operators $X_i^{\sigma 0}$'s project the electron creation operators onto a space spanned by the basis $\{|0\rangle_i, |\sigma\rangle_i\}$ and take the form $X_i^{\sigma 0} = |\sigma\rangle_i\langle 0|_i$. Together with the bosonic generators, $X_i^{\sigma\sigma'} = |\sigma\rangle_i\langle\sigma'|_i$ the full set of operators $X_i^{ab}$, $a,b = 0,\uparrow,\downarrow$ forms on each lattice site a basis for the fundamental representation of the semisimple doubly graded Lie algebra u(2|1) given by the (anti)commutation relations

$$\{X_i^{ab}, X_j^{cd}\}_\pm = (X_i^{ad}\delta^{bc} \pm X_j^{bc}\delta^{ad})\delta^{ij}\,,$$

where the (+) sign holds only when both operators are fermionic. In fact the identity

$$X^{00} + \sum_\sigma X^{\sigma\sigma} = 1$$

reduces this superalgebra to the eight-dimensional su(2|1) superalgebra.

Since su(2|1) can be viewed as a supergeneralization of the conventional spin su(2) algebra, the $t$–$J$ Hamiltonian appears as a superextension of the Heisenberg magnetic Hamiltonian, with a hole being a superpartner of a su(2) magnetic excitation [1]. This superalgebra can also be thought of as a natural generalization of the standard fermionic algebra spanned by generators $c_\sigma^+$, $c_\sigma$ and unity $I$ for the case in which the fermionic operators are subject to the NDO constraint. The incorporation of this constraint manifests itself in more complicated commutation relations between the $X$ operators in comparison with those produced by the conventional fermionic operators.

In order to get around the problem of dealing with rather complicated commutation relations of su(2|1), one in practice frequently uses the so-called "slave particle" representations of the Hubbard operators. In the Lie algebra language these representations correspond to the oscillator representations of the Lie algebra generators. For example, let $X_{\lambda\lambda'}$, $\lambda = 1,2,3$ be a matrix corresponding to the operator $X$. Consider a composite creation operator $d^\dagger = (a^\dagger, b^\dagger, f^\dagger)$, where $a$

and $b$ stand for bosonic field and $f$ for a fermionic one. Then, the slave-fermion representation reads

$$X = \sum_{\lambda\lambda'} d^{\dagger}_{\lambda} X_{\lambda\lambda'} d_{\lambda'}\,, \qquad \sum_{\lambda} d^{\dagger}_{\lambda} d_{\lambda} = a^{\dagger}a + b^{\dagger}b + f^{\dagger}f = 1\,,$$

where the last equation is a linear Casimir operator of u(2|1), whose lowest non-trivial eigenvalue equals to 1, fixes the lowest 3D representation spanned by the Hubbard operators.

Although one is dealing now with the standard bosonic and fermionic operators, equation $a^{\dagger}_i a_i + b^{\dagger}_i b_i + f^{\dagger}_i f_i = 1$ must hold at each lattice site, which poses a severe technical problem in practical calculations. Usually this local constraint is replaced by a global one, which however results in uncontrollable approximations.

The above technical problems motivates our present attempt at working directly within the su(2|1) superalgebra representation of the $t$–$J$ Hamiltonian. Since the $X$ operators are generators of the su(2|1) superalgebra we are led naturally to employ the su(2|1) coherent-state path-integral representation of the $t$–$J$ partition function. This provides a mathematical setting well adjusted to address the $t$–$J$ model with the NDO constraint naturally built in the formalism from the very beginning. This approach can also be derived starting from the slave fermion theory, as shown in Appendix.

## 3 su(2|1) coherent states and path integral

The normalized su(2|1) coherent state (CS) associated with the 3D fundamental representation takes the form

$$|z,\xi\rangle = \left(1+\bar{z}z+\bar{\xi}\xi\right)^{-1/2} \exp\left(zX^{\downarrow\uparrow} + \xi X^{0\uparrow}\right) |\uparrow\rangle\,, \tag{3}$$

where $z$ is a complex number and $\xi$ is a complex Grassmann parameter. The set $(z,\xi)$ can be thought of as local coordinates of a given point on $\mathrm{CP}^{1|1}$. This supermanifold appears as a $N=1$ superextension of a complex projective plane, or ordinary two-sphere, $\mathrm{CP}^1 = S^2$, to accommodate one extra complex Grassmann parameter [6]. At $\xi=0$ the su(2|1) CS reduces to the ordinary su(2) CS, $|z,\xi=0\rangle \equiv |z\rangle$ parametrized by a complex coordinate $z \in \mathrm{CP}^1$. Note that the classical phase space of the Hubbard operators, $\mathrm{CP}^{1|1}$, appears as a $N=1$ superextension of the CS manifold for the su(2) spins.

In the basis $|z,\xi\rangle = \prod_j |z_j,\xi_j\rangle$, the $t$–$J$ partition function takes the form of the su(2|1) CS phase-space path integral,

$$Z_{t-J} = \mathrm{tr}\exp(-\beta H_{t-J}) = \int_{\mathrm{CP}^{1|1}} D\mu_{\mathrm{SU}(2|1)}(z,\xi)\, \mathrm{e}^{S_{t-J}}\,, \tag{4}$$

where

$$D\mu_{\mathrm{SU}(2|1)}(z,\xi) = \prod_{j,t} \frac{\mathrm{d}\bar{z}_j(t)\mathrm{d}z_j(t)}{2\pi\mathrm{i}}\, \frac{\mathrm{d}\bar{\xi}_j(t)\mathrm{d}\xi_j(t)}{1+|z_j|^2+\bar{\xi}_j\xi_j}$$

stands for the SU(2|1) invariant measure with the boundary conditions, $z_j(0) = z_j(\beta)$, $\xi_j(0) = -\xi_j(\beta)$. The $t$–$J$ effective action on $\mathrm{CP}^{1|1}$ now reads

$$S_{t-J} = -\int_0^\beta \Big\langle z,\xi\Big|\frac{\mathrm{d}}{\mathrm{d}t} + H_{t-J}\Big|z,\xi\Big\rangle \mathrm{d}t\,,$$

which gives

$$S_{t-J} = \frac{1}{2}\sum_j \int_0^\beta \frac{\dot{\bar{z}}_j z_j - \bar{z}_j\dot{z}_j + \dot{\bar{\xi}}_j\xi_j - \bar{\xi}_j\dot{\xi}_j}{1+|z_j|^2+\bar{\xi}_j\xi_j}\,\mathrm{d}t - \int_0^\beta H^{\mathrm{cl}}_{t-J}\,\mathrm{d}t\,. \qquad (5)$$

The first part of the action (5) is a kinetic term that appears at each lattice site as an integral of the SU(2|1) symplectic one-form while the classical image of the Hamiltonian $H^{\mathrm{cl}}_{t-J} = \langle z,\xi|H_{t-J}|z,\xi\rangle$.

The Berezin covariant symbols of the classical observables corresponding to the supergenerators of SU(2|1) have already been evaluated in [6] and are given by

$$\begin{array}{llll} Q_3^{\mathrm{cl}} = -\frac{1}{2}\left(1-|z|^2\right)w\,, & (Q^+)^{\mathrm{cl}} = zw\,, & (Q^-)^{\mathrm{cl}} = \bar{z}w\,, & (X^{00})^{\mathrm{cl}} = \bar{\xi}\xi w\,, \\ (X^{0\downarrow})^{\mathrm{cl}} = -z\bar{\xi}w\,, & (X^{0\uparrow})^{\mathrm{cl}} = -\bar{\xi}w\,, & (X^{\uparrow 0})^{\mathrm{cl}} = -\xi w\,, & (X^{\downarrow})^{\mathrm{cl}} = -\bar{z}\xi w\,, \end{array}$$

where $w := \left(1+|z|^2+\bar{\xi}\xi\right)^{-1}$.

Upon making two successive changes of variables,

$$z \to z\sqrt{1+\bar{\xi}\xi}\,, \quad \xi \to \xi\sqrt{1+|z|^2}\,,$$

the SU(2|1) invariant measure and the kinetic term in eq. (4) are decoupled into the SU(2) invariant spinon and U(1) invariant fermion pieces. The corresponding transformations of the classical observables can easily be evaluated. In particular,

$$\begin{aligned} Q_3^{\mathrm{cl}} &\to -\frac{1}{2}\,\frac{1-|z|^2}{1+|z|^2} + \frac{1}{2}\,\bar{\xi}\xi = S_3^{\mathrm{cl}} + \frac{1}{2}\,\bar{\xi}\xi\,, \\ n_e^{\mathrm{cl}} \;&=\; X_{\mathrm{cl}}^{\uparrow\uparrow} + X_{\mathrm{cl}}^{\downarrow\downarrow} = \frac{1+|z|^2}{1+|z|^2+\bar{\xi}\xi} \to 1-\bar{\xi}\xi\,. \end{aligned}$$

This representation turns out to be useful to implement the RVB spin-charge separation and is used in the next section.

## 4 The boson−spinless fermion model

The origin of the pseudogap is one of the most important current problems in high temperature superconductors. The pseudogap opens at temperatures much higher than the superconducting transition temperature and shows up as a reduction of the density of states at the Fermi level. The pseudogap deepens with the lowering of the temperature and at the superconducting transition temperature smoothly evolves into the superconducting gap. This normal state gap is observed by various techniques, including angle-resolved photoemission spectroscopy [7], NMR [8], infrared [9], and transport [10] measurements. Although it is generally believed that the pseudogap reflects the strongly interacting regime of the electronic correlations [11], the understanding of this phenomenon is still far from complete.

The opening of the pseudogap, and its connection with the true superconducting gap is still a matter of debate. In one possible scenario the onset of superconductivity is controlled by phase fluctuations which continue to produce a finite pairing amplitude well above $T_c$. Within this framework the boson–fermion model (BFM) [12–15] has been successful in describing how the pseudogap evolves into the true superconducting gap. The Hamiltonian for the BFM is given by

$$\begin{aligned} H_{\mathrm{BFM}} = (\epsilon_0 - \mu) \sum_{i\sigma} c^{\dagger}_{i\sigma} c_{i\sigma} - t \sum_{\langle ij \rangle, \sigma} c^{\dagger}_{i\sigma} c_{j\sigma} + \\ + (E_0 - 2\mu) \sum_i b^{\dagger}_i b_i + g \sum_i [b^{\dagger}_i c_{i\downarrow} c_{i\uparrow} + b_i c^{\dagger}_{i\uparrow} c^{\dagger}_{j\downarrow}] \,. \end{aligned} \tag{6}$$

Here the $c^{(\dagger)}_{i\sigma}$'s denote annihilation (creation) operators for electrons with spin $\sigma$ at site $i$ and $b^{(\dagger)}_i$'s stand for bosonic operators describing tightly bound localized electron pairs. $E_0$, $\epsilon_0$ and $g$ are phenomenological parameters and the chemical potential $\mu$ is assumed to be common to the two kinds of fields in order to ensure charge conservation. This "standard" BFM is therefore understood as a system of localized tightly bound electron pairs (bosons) which hybridize with itinerant electrons. The BFM is usually introduced in a phenomenological level and the microscopic origin of the pair correlations remains unspecified. Recently an effective plaquette BFM was shown to be the low energy limit of a Hubbard model in a square lattice. This was achieved by applying a contractor renormalization method [16] to a plaquettized lattice to compute the boson effective interaction.

In this section we apply the su(2|1) path-integral technique to show that within the RVB–LSW approximation the $t$–$J$ Hamiltonian (2) may result in yet another variant of BFM composed of spinless fermions and bosonic spin waves. While the standard BFM consists of interacting electrons and bosons represented by tightly bound electron pairs of polaronic origin, the spin–charge separation inherent in the RVB phase naturally implies other types of elementary excitations. Within our approach spinless U(1) charged fermions are generated by the NDO constraint, while the bosonic fields correspond to chargeless spinon excitations describing SU(2) spin singlets in the LSW approximation. Therefore our model markedly contrasts with the standard BFM of itinerant spin-$\frac{1}{2}$ electrons and tightly bound electron pairs, and is therefore referred to as the boson–spinless fermion model (BSFM).

We start by formulating the RVB approximation to the $t$–$J$ Hamiltonian (2). This Hamiltonian possesses two global U(1) symmetries: $\mathrm{U}_{N_e}(1)$ and $\mathrm{U}_{Q_3}(1)$. These correspond to the conservation of the total electron number operator $N_e = \sum_i (X^{\uparrow\uparrow}_i + X^{\downarrow\downarrow}_i)$, and the total spin projection operator $Q_3 = \frac{1}{2} \sum_i (X^{\uparrow\uparrow}_i - X^{\downarrow\downarrow}_i)$, respectively.

A conventional Hartree–Fock decoupling is applied to the magnetic part of the $t$–$J$ Hamiltonian

$$H_J := J \sum_{\langle i,j\rangle} \left( \vec{Q}_i \vec{Q}_j - \tfrac{1}{4}\, n_i n_j \right) = -J \sum_{\langle i,j\rangle} b^\dagger_{ij} b_{ij}$$

yielding

$$H_J \simeq -J \sum_{\langle i,j\rangle} \left( \Delta_{ij} b^\dagger_{ij} + \Delta^*_{ij} b_{ij} - |\Delta_{ij}|^2 \right),$$

where $b^\dagger_{ij} = \frac{1}{\sqrt{2}} \left( X_i^{\uparrow 0} X_j^{\downarrow 0} - X_i^{\downarrow 0} X_j^{\uparrow 0} \right)$ is the valence bond "singlet" pair creation operator and $\Delta_{ij}$ is the RVB order parameter defined on each bond between the nearest neighbor sites. The condition $\Delta_{ij} \neq 0$ breaks the global $\mathrm{U}_{N_e}(1)$ symmetry, though it does not directly result in superconductivity. It instead indicates the onset of the electron spin–singlet formation.

The $t$–$J$ partition function becomes

$$\begin{aligned} Z_{t-J} &= \int \prod_j D\mu^{(j)}_{\mathrm{SU}(2)} \int \prod_j D\mu^{(j)}_{\mathrm{U}(1)} \exp[A_{t-J}], \\ A_{t-J} &= \frac{1}{2} \sum_j \int_0^\beta \left( \frac{\dot{\bar{z}}_j z_j - \bar{z}_j \dot{z}_j}{1+|z_j|^2} + \dot{\bar{\xi}}_j \xi_j - \bar{\xi}_j \dot{\xi}_j \right) \mathrm{d}t - \int_0^\beta H^{\mathrm{cl}}_{t-J} \mathrm{d}t \end{aligned} \tag{7}$$

with $\xi_i$ corresponding to the U(1) charged spinless fermion degrees of freedom (holons), and $z_i$ representing pure SU(2) spins (spinons). Here

$$D\mu_{\mathrm{SU}(2)} = \frac{\mathrm{d}\bar{z}\, \mathrm{d}z}{\pi \mathrm{i} \left(1+|z|^2\right)^2}$$

stands for the SU(2) invariant measure, while $D\mu_{\mathrm{U}(1)} = \mathrm{d}\bar{\xi}\, \mathrm{d}\xi$ denotes the Berezin integration over Grassmann variables.

The classical Hamiltonian, that enters the partition function (7), reads

$$\begin{aligned} H^{\mathrm{cl}}_{t-J} &= -t \sum_{\langle i,j\rangle} \xi_i \bar{\xi}_j \langle z_i | z_j \rangle - \frac{J\Delta_{\mathrm{RVB}}}{2} \sum_{\langle i,j\rangle} \xi_i \xi_j \bar{\Phi}_{ij} + \mathrm{h.c.} - \\ &\quad -\mu' \sum_i (1 - \bar{\xi}_i \xi_i) - \lambda \sum_i \left[ 2S_3^{\mathrm{cl}}(\bar{z}_i, z_i) + \bar{\xi}_i \xi_i \right]. \end{aligned} \tag{8}$$

Here we have dropped the constant term and explicitly introduced a chemical potential, $\mu'$, as well as the Lagrange multiplier $\lambda$ to control the number of electrons $N_e^{\mathrm{cl}} = \sum_i (1 - \bar{\xi}_i \xi_i)$ and the magnitude of the total electron magnetic moment $2Q_3^{\mathrm{cl}} = \sum_i (2S^{\mathrm{cl}}_{3i} + \bar{\xi}_i \xi_i)$, respectively. Note that the classical observables that correspond to the su(2) spin algebra generators are given by ($A^{\mathrm{cl}} := \langle z | A | z \rangle$)

$$S_3^{\mathrm{cl}} = -\frac{1}{2} \frac{1-|z|^2}{1+|z|^2}, \quad S_+^{\mathrm{cl}} = \frac{\bar{z}}{1+|z|^2}, \quad S_-^{\mathrm{cl}} = \frac{z}{1+|z|^2}.$$

In eq.(8) $|z\rangle$ stands for the su(2) coherent state and

$$\langle z_i|z_j\rangle = \frac{(1+\bar{z}_i z_j)}{\sqrt{(1+|z_i|^2)(1+|z_j|^2)}}$$

is related to the spinon–singlet amplitude

$$\Phi_{ij} = \frac{z_j - z_i}{\sqrt{(1+|z_i|^2)(1+|z_j|^2)}}$$

by the equation $|\langle z_i|z_j\rangle|^2 = 1 - |\Phi_{ij}|^2$. Explicitly, the spinon–singlet amplitude takes the form

$$\Phi_{ij} = \Psi_\downarrow(z_i)\Psi_\uparrow(z_j) - \Psi_\uparrow(z_i)\Psi_\downarrow(z_j)\,,$$

where $\Psi_{\uparrow\downarrow}(z)$ is a spinon wave function in the su(2) coherent state representation, $\Psi_{\uparrow\downarrow}(z) := \langle\uparrow\downarrow|z\rangle$.

The LSW approximation amounts to expanding up to leading order in powers of $|z|^2$ the action (7) as well as a measure factor in the path integral, which is justified in view of the fact that we are in the dilute limit of the spinon singlets, i.e. when $|\Phi_{ij}|^2 \ll 1$. The spinon singlets describe small quantum fluctuations of the spinon field that directly interact with holons. Those fluctuations arise around a classical spinon configuration ($z_i = 0$) that, however, contributes trivially to eq. (8) and the partition function. Since the term proportional to $\Delta_{\mathrm{RVB}}$ drops out of the Hamiltonian (8), this classical background physically corresponds to a gas of noninteracting spinless fermions with a modified chemical potential $\mu' \to \mu' - \lambda$.

In the paramagnetic phase $\langle Q_{3,i}\rangle = 0$, and the spinon fluctuations become bounded by the condition $\langle|z_i|^2\rangle \le \frac{1}{2}$. Within the LSW approximation the spinon–singlet amplitude reduces to $\Phi_{ij} = z_j - z_i$, and the partition function becomes

$$Z_{\mathrm{BSFM}} = \int \prod_j \mathrm{d}\bar{z}_j\,\mathrm{d}z_j\,\mathrm{d}\bar{\xi}_j\,\mathrm{d}\xi_j\,\exp[A_{\mathrm{BSFM}}]\,,$$

$$A_{\mathrm{BSFM}} = \frac{1}{2}\sum_j \int_0^\beta \left(\dot{\bar{z}}_j z_j - \bar{z}_j \dot{z}_j + \dot{\bar{\xi}}_j \xi_j - \bar{\xi}_j \dot{\xi}_j\right)\mathrm{d}t - \int_0^\beta H^{\mathrm{cl}}_{\mathrm{BSFM}}\mathrm{d}t\,,$$

where the classical Hamiltonian now reads

$$\begin{aligned} H^{\mathrm{cl}}_{\mathrm{BSFM}} = {} & -t\sum_{\langle i,j\rangle}\xi_i\bar{\xi}_j - \frac{J\Delta_{\mathrm{RVB}}}{2}\sum_{\langle i,j\rangle}\xi_i\xi_j(\bar{z}_j - \bar{z}_i) + \mathrm{h.c.} - \\ & -\mu'\sum_i(1-\bar{\xi}_i\xi_i) - \lambda\sum_i\left[-1 + 2\bar{z}_i z_i + \bar{\xi}_i\xi_i\right]. \end{aligned} \tag{9}$$

Let us at this stage clarify the physical meaning of the applied LSW approximation. Suppose we choose in representation (8) a classical spinon configuration with all $z_i = 0$. In this case the operator $2Q_{3i}$ reduces to $-N_{ei}$, so that Eq. (8) describes in this limit, as it should, a $\mathrm{U}_{N_e}(1)$ charged gas of noninteracting spinless fermions that exhibits no magnetic properties. Assuming it further that the interaction strength $J\Delta_{\rm RVB}/2 \ll t$, we consider the interaction term as a small perturbation. When $z_i$ becomes nonzero we can naturally arrive at the representation (9) that now describes a weak interaction regime for spinless fermions and a dilute gas of the su(2) spinon singlets at a fixed value of the total electron magnetic moment.

The above partition function represents the BSFM Hamiltonian

$$\begin{aligned} H_{BSFM} &= -t\sum_{\langle i,j\rangle} f_i f_j^\dagger - 2v\sum_{\langle i,j\rangle} f_i f_j (b_j^\dagger - b_i^\dagger) + \text{h.c.} + \\ &\quad + \Delta_B \sum_i b_i^\dagger b_i - \mu \sum_i (2b_i^\dagger b_i + f_i^\dagger f_i)\,, \\ \{f_i, f_j^\dagger\} &= [b_j, b_j^\dagger] = \delta_{ij}\,. \end{aligned} \tag{10}$$

The parameters of the boson–fermion system are expressed in terms of the parameters of the original $t$–$J$ Hamiltonian and the RVB state: $v = \frac{1}{4}J\Delta_{\rm RVB}$; $\Delta_B = 2\mu'$; $\mu = \mu' + \lambda$. In contrast to the standard BFM (6) the boson–fermion interaction term in (14) is a nonlocal function of the lattice sites. This is a consequence of the spinless character of fermions due to the NDO constraint. Elementary fermionic and bosonic excitations appear in this model as the RVB holons and spinon waves, respectively.

The $\mathrm{U}_{Q_3}(1)$ symmetry of eq. (8) under global gauge transformations $z_i \to e^{i\theta} z_i$, $\xi_i \to e^{i\theta/2}\xi_i$ reduces to the conservation of the projection of the total electron magnetic moment $2Q_3 = \sum_i (2b_i^\dagger b_i + f_i^\dagger f_i - 1)$ in eq. (10). A global U(1) symmetry characterizes the standard BFM (6) indicating that bosons are doubly charged with respect to fermions of a charge $e$. However, within the BSFM approach the role of the charge $e$ is played by the lowest eigenvalue of $Q_3$, $s = \frac{1}{2}$. The bosonic $b$ field is doubly "$s$"-charged with respect to the fermionic $f$ field. Since a long-range magnetic order is excluded, the local boson and fermion particle densities are related in our calculations by the condition $2n_B + n_F - 1 = 0$. In contrast to the conventional BFM, the boson and fermion excitations bear now information on the magnetic properties of the system, rather than on a charge transport.

## 5 Conclusions

To conclude, we employ the su(2|1) coherent-state path integral representation of the $t$–$J$ partition function to describe strongly correlated lattice electron system relevant for high-$T_c$ superconductivity. Strong correlation manifest itself by the local NDO constraint which is taken into account rigorously in this approach. The effective action is a function of local coordinates on the SU(2|1) homogeneous coherent-state supermanifold, $\mathrm{CP}^{1|1}$, which represents the phase space of the Hubbard operators.

To illustrate a possible application of this technique we show that within the RVB and LSW approximations this effective action reduces to that of the boson–spinless fermion model. Spinless fermions appear due to the NDO constraint.

We express the parameters of the effective BSFM Hamiltonian in terms of those of the $t$–$J$ model and the RVB state. The spin–charge separation inherent in the RVB phase results in a specific structure of the BSFM: it describes spinless fermions interacting with the dilute gas of the spinon singlets in such a way that the total electron spin projection which plays in the LSW approximation the role of the total effective charge is conserved. Numerical calculations [17] show that a local minimum in the fermionic density of states occurs similar to the standard BFM close to the Fermi level. The minimum deepens with the decreasing of temperature and vanishes when the temperature is sufficiently high.

### Appendix: su(2|1) path integral vs slave fermion approach

The fact that the electron system with the NDO constraint lives on the compact manifold, supersphere $CP^{1|1}$ can be explained as follows [3]. Let us for a moment suppose that the so-called slave-fermion representation for the electron operators is used, i.e.

$$c_{i\sigma} = f_i a^+_{i\sigma}\,, \tag{11}$$

where $f_i$ is a on-site spinless fermionic operator, whereas $a_{i\sigma}$ is the spinful boson. The NDO constraint now reads $\sum_\sigma a^+_{i\sigma} a_{i\sigma} + f^+_i f_i = 1$. Within the slave-fermion path integral representation

$$Z_{t-J} = \int D\mu_{\rm flat}\, {\rm e}^{S_{t-J}(\overline{a}_\sigma, a_\sigma, f)}, \tag{12}$$

with the integration measure $D\mu_{\rm flat} = \prod_i D\overline{a}_{i\uparrow} Da_{i\uparrow} D\overline{a}_{i\downarrow} Da_{i\downarrow} D\overline{f_i} Df_i$, this constraint transforms into

$$\sum_\sigma \overline{a}_{i\sigma} a_{i\sigma} + \overline{f}_i f_i = 1\,, \tag{13}$$

with $a_{i\sigma}$ and $f_i$ standing now for complex numbers and complex Grassmann parameters, respectively. Equation (13) is exactly that for the supersphere $CP^{1|1}$ embedded into a flat superspase. Any mean-field treatment of (12) should respect this constraint, which, however, poses a severe technical problem. If one however resolves this equation explicitly by making the identifications

$$\begin{aligned} a_{i\uparrow} &= \frac{{\rm e}^{{\rm i}\phi_i}}{\sqrt{1+\overline{z}_i z_i + \overline{\xi}_i \xi_i}}\,, \\ a_{i\downarrow} &= \frac{z_i {\rm e}^{{\rm i}\phi_i}}{\sqrt{1+\overline{z}_i z_i + \overline{\xi}_i \xi_i}}\,, \\ f_i &= \frac{\xi_i {\rm e}^{{\rm i}\phi_i}}{\sqrt{1+\overline{z}_i z_i + \overline{\xi}_i \xi_i}}\,, \end{aligned} \tag{14}$$

one can further treat the variables $z_i$, $\xi_i$ as if they were indeed free of any constraints.

Note that the electron operator (11) is invariant under a local gauge transformation,

$$a_{i\sigma} \to a_{i\sigma} \mathrm{e}^{\mathrm{i}\theta_i}\,, \quad f_i \to f_i \mathrm{e}^{\mathrm{i}\theta_i}\,,$$

which is tantamount to taking $\phi_i \to \phi_i + \theta_i$. This additional local gauge symmetry is a consequence of the redundancy of parameterizing the electron operator in terms of the auxiliary boson/fermion fields. In contrast, the su(2—1) projected coordinates

$$z_i = a_{i\downarrow}/a_{i\uparrow}, \quad \xi_i = f_i/a_{i\uparrow}$$

are seen to be manifestly gauge invariant.

Evaluating the superdeterminant

$$\mathrm{sdet}\left\| \frac{\partial(a_{\downarrow}, \bar{a}_{\downarrow}, f, \bar{f})}{\partial(z, \bar{z}, \xi, \bar{\xi})} \right\| = \frac{1}{1+|z|^2+\bar{\xi}\xi}$$

and substituting of (14) into (12) we are led to the su(2|1) path-integral representation of $Z_{t-J}$ given by Eq. (4). Note that the U(1) gauge field $\phi_i$ drops out of the representation (4).

## References


[1] P.B. Wiegmann: Phys. Rev. Lett. **60** (1988) 821.

[2] A Foerster and M. Karowski: Nucl. Phys. B **396** (1993) 611.

[3] A. Ferraz, E. Kochetov and M. Mierzejewski: `cond--mat/0412235`.

[4] K.A. Chao, J. Spałek and A.M. Oleś: Phys. Rev. B **18** (1978)3453; R. Gros, R. Joint and T.M. Rice: Phys. Rev. B **36** (1987) 8190.

[5] J. Hubbard: Proc. R. Soc. London Ser. A **285** (1965) 542.

[6] E. Kochetov and V. Yarunin: Phys. Rev. B **56** (1997) 1580.

[7] D.S. Marshall, D.S. Dessau, A.G. Loeser, C-H. Park, A.Y. Matsuura, J.N. Eckstein, I. Bozovic, P. Fournier, A. Kapitulnik, W.E. Spicer and Z-X. Shen: Phys. Rev. Lett. **76** (1996) 4841.

[8] G.V.M. Williams, J.L. Tallon, E.M. Haines, R. Michalak and R. Dupree: Phys. Rev. Lett. **78** (1997) 721.

[9] D.N. Basov, T. Timusk, B. Dabrowski and J.D. Jorgensen: Phys. Rev. B **50** (1994) 3511.

[10] J.L. Tallon, J.R. Cooper, P.S.I.P.N de Silva, G.V.M. Williams and J.W. Loram: Phys. Rev. Lett. **75** (1995) 4114.

[11] E.W. Carlson, V.J. Emery, S.A. Kivelson and D. Orgad: `cond-mat/0206217`.

[12] S. Robaszkiewicz, R. Micnas and J. Ranninger: Phys. Rev. B **36** (1987) 180.

[13] J. Ranninger, J.M. Robin and M. Eschrig: Phys. Rev. Lett. **74** (1995) 4027.

[14] P. Devillard and J. Ranninger: Phys. Rev. Lett. **84** (2000) 5200.

[15] T. Domański and J. Ranninger: Phys. Rev. B **63** (2001) 134505.

[16] E. Altman and A. Auerbach: Phys. Rev.B **65** (2002) 104508.

[17] A. Ferraz, E. Kochetov and M. Mierzejewski: to be published.