

Smooth functional derivatives in Feynman path integrals

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This note is an exposition of our recent papers [10, 23]. We give a fairly general class of functionals on a path space so that Feynman path integral has a mathematically rigorous meaning. More precisely, for any functional belonging to our class, the time slicing approximation of Feynman path integral converges uniformly on compact subsets of the configuration space. Our class of functionals is closed under addition, multiplication, translation, real linear transformation and functional differentiation. The integration by parts and Taylor's expansion formula with respect to functional differentiation holds in Feynman path integral. Feynman path integral is invariant under translation and orthogonal transformation. The interchange of the order with Riemann–Stieltjes integrals, the interchange of the order with a limit, the perturbation expansion formula, the semiclassical approximation and the fundamental theorem of calculus holds in Feynman path integral.

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1 Introduction

In 1948, R.P. Feynman [3] expressed the integral kernel of the fundamental solution for the Schrödinger equation, using the path integral as follows:

$$\int e^{iS[\gamma]/\hbar} \mathcal{D}[\gamma]. \quad (1.1)$$

Here $0 < \hbar < 1$ is Planck's constant, $\gamma : [0, T] \rightarrow \mathbf{R}^d$ is a path with $\gamma(0) = x_0$ and $\gamma(T) = x$, and $S[\gamma]$ is the action along the path γ defined by

$$S[\gamma] = \int_0^T \left(\frac{1}{2} \left| \frac{d\gamma}{dt} \right|^2 - V(t, \gamma(t)) \right) dt. \quad (1.2)$$

The path integral is a new sum of $e^{iS[\gamma]/\hbar}$ over all the paths. Feynman explained his new integral as a limit of the finite dimensional integral, which is now called

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the time slicing approximation. Furthermore, Feynman suggested a new analysis on a path space with the functional integration

$$\int F[\gamma]e^{iS[\gamma]/\hbar}\mathcal{D}[\gamma],$$

and the functional differentiation $(DF)[\gamma][\eta]$. (cf. Feynman–Hibbs [4], L.S. Schulman [27]) However, in 1960, R.H. Cameron [2] proved that the completely additive measure $e^{iS[\gamma]/\hbar}\mathcal{D}[\gamma]$ does not exist.

In this note, using the time slicing approximation, we treat the functional differentiation $(DF)[\gamma][\eta]$, the translation $F[\gamma + \eta]$ and the orthogonal transformation $F[Q\gamma]$ in the Feynman path integral

$$\int e^{iS[\gamma]/\hbar}F[\gamma]\mathcal{D}[\gamma] \tag{1.3}$$

for a fairly general class \mathcal{F}^∞ of functionals $F[\gamma]$ on the path space $C([0, T] \rightarrow \mathbf{R}^d)$ by a mathematically rigorous discussion.

Our class $F[\gamma] \in \mathcal{F}^\infty$ satisfies the following:

- (0) For any $F[\gamma] \in \mathcal{F}^\infty$, (1.3) has a mathematically rigorous meaning. More precisely, the time slicing approximation of (1.3) converges uniformly on any compact set on the configuration space \mathbf{R}^{2d} with respect to endpoints (x, x_0) .
- (1) The class \mathcal{F}^∞ is an algebra, i.e.

$$F[\gamma], G[\gamma] \in \mathcal{F}^\infty \implies F[\gamma] + G[\gamma], F[\gamma]G[\gamma] \in \mathcal{F}^\infty.$$

- (2) The class \mathcal{F}^∞ is closed under the translation with any broken line path η and the linear transformation with any $d \times d$ real matrix P , i.e.

$$F[\gamma] \in \mathcal{F}^\infty \implies F[\gamma + \eta], F[P\gamma] \in \mathcal{F}^\infty.$$

- (3) The class \mathcal{F}^∞ is closed under the functional differentiation D , i.e.

$$F[\gamma] \in \mathcal{F}^\infty \implies (DF)[\gamma][\eta] \in \mathcal{F}^\infty.$$

In other words, we can differentiate $F[\gamma]$ as many times as we want.

Furthermore, the class \mathcal{F}^∞ contains the following examples of functionals $F[\gamma]$:

- (a) For any function B such that $|\partial_x^\alpha B(t, x)| \leq C_\alpha(1 + |x|)^m$ for some $m > 0$ and any $0 \leq t \leq T$, the evaluation functional

$$F[\gamma] = B(t, \gamma(t)) \in \mathcal{F}^\infty.$$

In particular, if $F[\gamma] \equiv c \in \mathbf{C}$, then $F[\gamma] \in \mathcal{F}^\infty$.

(b) For the same B as in (a), the Riemann–Stieltjes integral

$$F[\gamma] = \int_{T'}^{T''} B(t, \gamma(t)) d\rho(t) \in \mathcal{F}^\infty.$$

(c) For any function B such that $|\partial_x^\alpha B(t, x)| \leq C_\alpha$, the analytic function of the integral

$$F[\gamma] = f \left(\int_{T'}^{T''} B(t, \gamma(t)) d\rho(t) \right) \in \mathcal{F}^\infty.$$

(d) For any vector-valued function Z such that $(\partial_x Z)$ is a symmetric matrix and $|\partial_x^\alpha Z(t, x)| + |\partial_x^\alpha \partial_t Z(t, x)| \leq C_\alpha (1 + |x|)^m$ for some $m > 0$, the curvilinear integral along paths

$$F[\gamma] = \int_{T'}^{T''} Z(t, \gamma(t)) \cdot d\gamma(t) \in \mathcal{F}^\infty.$$

Applying (1), (2), (3) to the examples (a), (b), (c), (d), the readers can produce many functionals $F[\gamma]$ belonging to the class \mathcal{F}^∞ .

As an application, the integration by parts and the Taylor expansion formula, with respect to functional differentiation holds in Feynman path integral. Furthermore, Feynman path integral is invariant under translation and orthogonal transformation. The interchange of the order with Riemann–Stieltjes integrals, the interchange of the order with a limit, the semiclassical approximation, and the fundamental theorem of calculus holds in Feynman path integral.

There are some mathematical works which prove the time slicing approximation of (1.1) converges uniformly with respect to endpoints (x, x_0) on compact subsets of \mathbf{R}^{2d} . See D. Fujiwara [5, 7–9], H. Kitada and H. Kumano-go [18], K. Yajima [29], N. Kumano-go [21], D. Fujiwara and T. Tsuchida [14], and W. Ichinose [15]. However all these works treated (1.1), that is the particular case of (1.3) with $F[\gamma] \equiv 1$.

Many people tried to give a mathematically rigorous meaning to Feynman path integral. E. Nelson [25] succeeded in connecting Feynman path integral to Wiener measure by analytic continuation with respect to a parameter. K. Itô [17] succeeded in defining Feynman path integrals as an improper oscillatory integral over a Hilbert manifold of paths. Albeverio and Høegh Krohn [1] and J. Rezende [26] applied Itô's idea and discussed many problems.

2 Main results

Let $\Delta_{T,0}$ be an arbitrary division of the interval $[0, T]$ into subintervals, i.e.

$$\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0. \quad (2.1)$$

Set $x_{J+1} = x$. Let $x_j, j = 1, 2, \dots, J$ be arbitrary points of \mathbf{R}^d . Let

$$\gamma_{\Delta_{T,0}} = \gamma_{\Delta_{T,0}}(t, x_{J+1}, x_J, \dots, x_1, x_0) \quad (2.2)$$

be the broken line path which connects (T_j, x_j) and (T_{j-1}, x_{j-1}) by a line segment for any $j = 1, 2, \dots, J, J+1$. Set $t_j = T_j - T_{j-1}$ and $|\Delta_{T,0}| = \max_{1 \leq j \leq J+1} t_j$.

As Feynman [3] had first defined by the time slicing approximation, we define the Feynman path integrals (1.3) by

$$\int e^{iS[\gamma]/\hbar} F[\gamma] \mathcal{D}[\gamma] = \lim_{|\Delta_{T,0}| \rightarrow 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{iS[\gamma_{\Delta_{T,0}}]/\hbar} F[\gamma_{\Delta_{T,0}}] \prod_{j=1}^J dx_j, \quad (2.3)$$

whenever the limit exists.

Remark. $S[\gamma_{\Delta_{T,0}}]$ and $F[\gamma_{\Delta_{T,0}}]$ are functions of a finite number of variables $x_{J+1}, x_J, \dots, x_1, x_0$, i.e.

$$\begin{aligned} S[\gamma_{\Delta_{T,0}}] &= S_{\Delta_{T,0}}(x_{J+1}, x_J, \dots, x_1, x_0), \\ F[\gamma_{\Delta_{T,0}}] &= F_{\Delta_{T,0}}(x_{J+1}, x_J, \dots, x_1, x_0). \end{aligned}$$

Therefore Feynman omitted the first step $S[\gamma_{\Delta_{T,0}}], F[\gamma_{\Delta_{T,0}}]$. Furthermore, many books about Feynman path integrals abandon the first step $S[\gamma_{\Delta_{T,0}}]$ in order to use the Trotter formula, i.e.

$$\begin{aligned} S[\gamma_{\Delta_{T,0}}] &= \sum_{j=1}^{J+1} \frac{(x_j - x_{j-1})^2}{2t_j} - \sum_{j=1}^{J+1} \int_{T_{j-1}}^{T_j} V \left(t, \frac{t - T_{j-1}}{T_j - T_{j-1}} x_j + \frac{T_j - t}{T_j - T_{j-1}} x_{j-1} \right) dt \\ &\neq \sum_{j=1}^{J+1} \frac{(x_j - x_{j-1})^2}{2t_j} - \sum_{j=1}^{J+1} V(T_{j-1}, x_{j-1}). \end{aligned}$$

However, we keep the first step $S[\gamma_{\Delta_{T,0}}], F[\gamma_{\Delta_{T,0}}]$.

Remark. Even when $F[\gamma] \equiv 1$, the integrals of the right hand side of (2.3) does not converge absolutely. We treat integrals of this type as oscillatory integrals. (cf. H. Kumano-go [19], H. Kumano-go and K. Taniguchi [20], D. Fujiwara, N. Kumano-go and K. Taniguchi [13], N. Kumano-go [22, 23])

Remark. If $|\Delta_{T,0}| \rightarrow 0$, the number J of the integrals of the right hand side of (2.3) tends to ∞ . Therefore, we use the properties of $F[\gamma_{\Delta_{T,0}}]$.

Remark. If we need the endpoints (x, x_0) , we will use the following expression:

$$\int_{\gamma(0)=x_0, \gamma(T)=x} e^{iS[\gamma]/\hbar} F[\gamma] \mathcal{D}[\gamma] = \int e^{iS[\gamma]/\hbar} F[\gamma] \mathcal{D}[\gamma].$$

Our assumption of the potential $V(t, x)$ of (1.2) is the following:

Assumption 1. (Potential). $V(t, x)$ is a real-valued function of $(t, x) \in \mathbf{R} \times \mathbf{R}^d$, and, for any multi-index α , $\partial_x^\alpha V(t, x)$ is continuous in $\mathbf{R} \times \mathbf{R}^d$. For any integer $k \geq 2$, there exists a positive constant A_k such that for any multi-index α with $|\alpha| = k$

$$|\partial_x^\alpha V(t, x)| \leq A_k.$$

In order to state the definition of the class \mathcal{F}^∞ of functionals $F[\gamma]$, we explain the functional derivatives in this note.

Definition 1. (Functional derivatives). *For any division $\Delta_{T,0}$, we assume that*

$$F_{\Delta_{T,0}}(x_{J+1}, x_J, \dots, x_1, x_0) \in C^\infty(\mathbf{R}^{d(J+2)}).$$

Let $\gamma : [0, T] \rightarrow \mathbf{R}^d$ and $\eta_l : [0, T] \rightarrow \mathbf{R}^d$, $l = 1, 2, \dots, L$ be any broken line paths. We define the functional derivative $(D^L F)[\gamma] \prod_{l=1}^L [\eta_l]$ by

$$(D^L F)[\gamma] \prod_{l=1}^L [\eta_l] = \left(\prod_{l=1}^L \frac{\partial}{\partial \theta_l} \right) F \left[\gamma + \sum_{l=1}^L \theta_l \eta_l \right] \Big|_{\theta_1 = \theta_2 = \dots = \theta_L = 0}.$$

When $L = 0$, we also write $(D^L F)[\gamma] \prod_{l=1}^L [\eta_l] = F[\gamma]$.

Remark. Let $\Delta_{T,0}$ of (2.1) contain all time where the broken line path γ or the broken line path η breaks. Set $\gamma(T_j) = x_j$ and $\eta(T_j) = y_j$, $j = 0, 1, \dots, J, J+1$. Then, for any $\theta \in \mathbf{R}$, $\gamma + \theta\eta$ is the broken line path, which connects $(T_j, x_j + \theta y_j)$ and $(T_{j-1}, x_{j-1} + \theta y_{j-1})$ by a line segment for $j = 1, 2, \dots, J, J+1$. Hence we have

$$F[\gamma + \theta\eta] = F_{\Delta_{T,0}}(x_{J+1} + \theta y_{J+1}, x_J + \theta y_J, \dots, x_1 + \theta y_1, x_0 + \theta y_0). \quad (2.4)$$

Therefore, we can write $(DF)[\gamma][\eta]$ as a finite sum as follows:

$$(DF)[\gamma][\eta] = \frac{d}{d\theta} F[\gamma + \theta\eta] \Big|_{\theta=0} = \sum_{j=0}^{J+1} (\partial_{x_j} F_{\Delta_{T,0}})(x_{J+1}, x_J, \dots, x_1, x_0) \cdot y_j. \quad (2.5)$$

Using (2.5) as an approximation, Feynman–Hibbs [4] explained functional derivatives. On the other hand, we ‘restrict’ the direction of functional derivatives to broken line paths. (cf. Malliavin’s derivatives [24].)

Definition 2. (The class \mathcal{F} of functionals $F[\gamma]$). *Let $F[\gamma]$ be a functional on the path space $C([0, T] \rightarrow \mathbf{R}^d)$ such that the domain of $F[\gamma]$ contains all of broken line paths at least. We say that $F[\gamma]$ belongs to the class \mathcal{F}^∞ if $F[\gamma]$ satisfies Assumption 2. For simplicity, we write $F[\gamma] \in \mathcal{F}^\infty$.*

Assumption 2. *Let m be a non-negative integer and $\rho(t)$ be a function of bounded variation on $[0, T]$. For any non-negative integer M , there exists a positive constant C_M such that*

$$\left| \left(D^{\sum_{j=0}^{J+1} L_j} F \right) [\gamma] \prod_{j=0}^{J+1} \prod_{l_j=1}^{L_j} [\eta_{j,l_j}] \right| \leq (C_M)^{J+2} (1 + \|\gamma\|)^m \prod_{j=0}^{J+1} \prod_{l_j=1}^{L_j} \|\eta_{j,l_j}\|, \quad (2.6)$$

$$\begin{aligned} & \left| \left(D^{1+\sum_{j=0}^{J+1} L_j} F \right) [\gamma][\eta] \prod_{j=0}^{J+1} \prod_{l_j=1}^{L_j} [\eta_{j,l_j}] \right| \\ & \leq (C_M)^{J+2} (1 + \|\gamma\|)^m \int_0^T |\eta(t)| d|\rho|(t) \prod_{j=0}^{J+1} \prod_{l_j=1}^{L_j} \|\eta_{j,l_j}\|, \end{aligned} \quad (2.7)$$

for any division $\Delta_{T,0}$ defined by (2.1), any $L_j = 0, 1, \dots, M$, any broken line path $\gamma : [0, T] \rightarrow \mathbf{R}^d$, any broken line path $\eta : [0, T] \rightarrow \mathbf{R}^d$, and any broken line paths $\eta_{j,l_j} : [0, T] \rightarrow \mathbf{R}^d$, $l_j = 1, 2, \dots, L_j$ whose supports exist in $[T_{j-1}, T_{j+1}]$. Here $0 = T_{-1} = T_0$, $T_{J+1} = T_{J+2} = T$, $\|\gamma\| = \max_{0 \leq t \leq T} |\gamma(t)|$ and $|\rho|(t)$ is the total variation of $\rho(t)$.

Remark. Note that the support of the broken line path η_{j,l_j} exists in $[T_{j-1}, T_{j+1}]$ for any $j = 0, 1, \dots, J, J+1$. Roughly speaking, the broken line paths η_{j,l_j} , $j = 0, 1, \dots, J, J+1$ slice the time interval $[0, T]$.

Remark. About the process how we were making up Assumption 2, see D. Fujiwara [6], N. Kumano-go [23] and D. Fujiwara, N. Kumano-go [10] in this order.

Theorem 1. (Existence of Feynman path integral). *Let T be sufficiently small. Then, for any $F[\gamma] \in \mathcal{F}^\infty$, the right hand side of (2.3) converges uniformly on any compact set of the configuration space $(x, x_0) \in \mathbf{R}^{2d}$, together with all its derivatives in x and x_0 .*

Remark. The size of T depends only on d and A_k of Assumption 1.

Theorem 2. (Smooth algebra). *For any $F[\gamma], G[\gamma] \in \mathcal{F}^\infty$, any broken line path $\zeta : [0, T] \rightarrow \mathbf{R}^d$ and any real $d \times d$ matrix P , we have the following.*

- (1) $F[\gamma] + G[\gamma] \in \mathcal{F}^\infty$, $F[\gamma]G[\gamma] \in \mathcal{F}^\infty$.
- (2) $F[\gamma + \zeta] \in \mathcal{F}^\infty$, $F[P\gamma] \in \mathcal{F}^\infty$.
- (3) $(DF)[\gamma][\zeta] \in \mathcal{F}^\infty$.

Remark. \mathcal{F}^∞ is closed under addition, multiplication, translation, real linear transformation and functional differentiation. Applying Theorem 2 to the examples of Theorem 3 (1), (2), Theorem 4 (1), Theorem 6 and Theorem 8, the reader can produce many functionals $F[\gamma] \in \mathcal{F}^\infty$.

Assumption 3. *Let m be a non-negative integer. $B(t, x)$ is a function of $(t, x) \in \mathbf{R} \times \mathbf{R}^d$. For any multi-index α , $\partial_x^\alpha B(t, x)$ is continuous on $\mathbf{R} \times \mathbf{R}^d$, and there exists a positive constant C_α such that*

$$|\partial_x^\alpha B(t, x)| \leq C_\alpha (1 + |x|)^m.$$

Theorem 3. (Interchange with Riemann-Stieltjes integrals). *Let $0 \leq T' \leq T'' \leq T$ and $0 \leq t \leq T$. Let $\rho(t)$ be a function of bounded variation on $[T', T'']$. Suppose $B(t, x)$ satisfy Assumption 3. Then we have the following:*

- (1) *The value at a fixed time t*

$$F[\gamma] = B(t, \gamma(t)) \in \mathcal{F}^\infty.$$

- (2) *The Riemann–Stieltjes integral*

$$F[\gamma] = \int_{T'}^{T''} B(t, \gamma(t)) d\rho(t) \in \mathcal{F}^\infty.$$

(3) Let T be sufficiently small. Then we have

$$\begin{aligned} & \int_{T'}^{T''} \left(\int e^{iS[\gamma]/\hbar} B(t, \gamma(t)) \mathcal{D}[\gamma] \right) d\rho(t) \\ &= \int e^{iS[\gamma]/\hbar} \left(\int_{T'}^{T''} B(t, \gamma(t)) d\rho(t) \right) \mathcal{D}[\gamma]. \end{aligned}$$

Furthermore, for any $F[\gamma] \in \mathcal{F}^\infty$, we have

$$\begin{aligned} & \int_{T'}^{T''} \left(\int e^{iS[\gamma]/\hbar} B(t, \gamma(t)) F[\gamma] \mathcal{D}[\gamma] \right) d\rho(t) \\ &= \int e^{iS[\gamma]/\hbar} \left(\int_{T'}^{T''} B(t, \gamma(t)) d\rho(t) \right) F[\gamma] \mathcal{D}[\gamma]. \end{aligned}$$

Remark. We explain the key of the proof of Theorem 3 (3) roughly. In order to use the Trotter formula, many books about Feynman path integrals approximate the position of the particle at time t by the endpoint x_j or x_{j-1} . On the other hand, using the number j so that $T_{j-1} < t \leq T_j$, we keep the position of the particle at time t , i.e.

$$\gamma_{\Delta_{T,0}}(t) = \frac{t - T_{j-1}}{T_j - T_{j-1}} x_j + \frac{T_j - t}{T_j - T_{j-1}} x_{j-1},$$

inside the finite dimensional oscillatory integral of (2.3). Therefore, we can use the continuity of the broken line path $\gamma_{\Delta_{T,0}}(t)$ with respect to t .

Proof of Theorem 3 (3).

(1) Note that $B(t, \gamma_{\Delta_{T,0}}(t))$ is a continuous function of t on $[T', T'']$ together with all its derivatives in x_j , $j = 0, 1, \dots, J, J+1$.

(2) By Lebesgue's dominated convergence theorem after integrating by parts by x_j , $j = 1, 2, \dots, J$ (Oscillatory integrals), for any division $\Delta_{T,0}$,

$$\prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{iS[\gamma_{\Delta_{T,0}}]/\hbar} B(t, \gamma_{\Delta_{T,0}}(t)) \prod_{j=1}^J dx_j$$

is also a continuous function of t on $[T', T'']$.

(3) By Theorem 1, the convergence of

$$\begin{aligned} & \int e^{iS[\gamma]/\hbar} B(t, \gamma(t)) \mathcal{D}[\gamma] \\ &= \lim_{|\Delta_{T,0}| \rightarrow 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{iS[\gamma_{\Delta_{T,0}}]/\hbar} B(t, \gamma_{\Delta_{T,0}}(t)) \prod_{j=1}^J dx_j \end{aligned}$$

is uniform with respect to t on $[T', T'']$.

(4) Therefore,

$$\int e^{iS[\gamma]/\hbar} B(t, \gamma(t)) \mathcal{D}[\gamma]$$

is also a continuous function of t on $[T', T'']$ and Riemann–Stieltjes integrable.
(5) Furthermore, by the uniform convergence, we can interchange the order of $\int_{T'}^{T''} \cdots dt$ and $\lim_{|\Delta_{T,0}| \rightarrow 0}$.

$$\begin{aligned} & \int_{T'}^{T''} \left(\int e^{iS[\gamma]/\hbar} B(t, \gamma(t)) \mathcal{D}[\gamma] \right) dt \\ &= \int_{T'}^{T''} \lim_{|\Delta_{T,0}| \rightarrow 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{iS[\gamma_{\Delta_{T,0}}]/\hbar} B(t, \gamma_{\Delta_{T,0}}(t)) \prod_{j=1}^J dx_j dt \\ &= \lim_{|\Delta_{T,0}| \rightarrow 0} \int_{T'}^{T''} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{iS[\gamma_{\Delta_{T,0}}]/\hbar} B(t, \gamma_{\Delta_{T,0}}(t)) \prod_{j=1}^J dx_j dt. \end{aligned}$$

By Fubini's theorem after integrating by parts by x_j , $j = 1, 2, \dots, J$ (Oscillatory integrals), we have

$$\begin{aligned} & \int_{T'}^{T''} \left(\int e^{iS[\gamma]/\hbar} B(t, \gamma(t)) \mathcal{D}[\gamma] \right) dt \\ &= \lim_{|\Delta_{T,0}| \rightarrow 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{iS[\gamma_{\Delta_{T,0}}]/\hbar} \int_{T'}^{T''} B(t, \gamma_{\Delta_{T,0}}(t)) dt \prod_{j=1}^J dx_j \\ &= \int e^{iS[\gamma]/\hbar} \left(\int_{T'}^{T''} B(t, \gamma(t)) dt \right) \mathcal{D}[\gamma]. \quad \square \end{aligned}$$

Assumption 4. $f(b)$ is an analytic function of $b \in \mathbf{C}$ on a neighborhood of zero, i.e., there exist positive constants $\mu > 0$, $A > 0$ such that

$$\|f\|_{\mu, A} = \sup_{n, |b| \leq \mu} \frac{|\partial_b^n f(b)|}{A^n n!} < \infty.$$

Theorem 4. (Interchange with a limit). Let $0 \leq T' \leq T'' \leq T$. Let $\rho(t)$ be a function of bounded variation on $[T', T'']$. Suppose $B(t, x)$ satisfy Assumption 3 with $m = 0$. Let $f(b)$ and $f_k(b)$, $k = 1, 2, 3, \dots$ be analytic functions such that $\lim_{k \rightarrow \infty} \|f_k - f\|_{\mu, A} = 0$. Then we have the following.

(1)

$$F[\gamma] = f \left(\int_{T'}^{T''} B(t, \gamma(t)) d\rho(t) \right) \in \mathcal{F}^\infty.$$

(2) Let T be sufficiently small. Then we have

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int e^{iS[\gamma]/\hbar} f_k \left(\int_{T'}^{T''} B(t, \gamma(t)) d\rho(t) \right) \mathcal{D}[\gamma] \\ &= \int e^{iS[\gamma]/\hbar} f \left(\int_{T'}^{T''} B(t, \gamma(t)) d\rho(t) \right) \mathcal{D}[\gamma]. \end{aligned}$$

Furthermore, for any $F[\gamma] \in \mathcal{F}^\infty$, we have

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int e^{iS[\gamma]/\hbar} f_k \left(\int_{T'}^{T''} B(t, \gamma(t)) d\rho(t) \right) F[\gamma] \mathcal{D}[\gamma] \\ &= \int e^{iS[\gamma]/\hbar} f \left(\int_{T'}^{T''} B(t, \gamma(t)) d\rho(t) \right) F[\gamma] \mathcal{D}[\gamma]. \end{aligned}$$

Corollary. (Perturbation expansion formula). Let T be sufficiently small. Let $\rho(t)$ and $B(t, x)$ be the same as in Theorem 4. Then we have

$$\begin{aligned} & \int \exp \left(\frac{i}{\hbar} S[\gamma] + \frac{i}{\hbar} \int_{T'}^{T''} B(\tau, \gamma(\tau)) d\rho(\tau) \right) \mathcal{D}[\gamma] \\ &= \sum_{n=1}^{\infty} \left(\frac{i}{\hbar} \right)^n \int_{T'}^{T''} d\rho(\tau_n) \int_{T'}^{\tau_n} d\rho(\tau_{n-1}) \cdots \int_{T'}^{\tau_2} d\rho(\tau_1) \\ & \quad \times \int e^{iS[\gamma]/\hbar} B(\tau_n, \gamma(\tau_n)) B(\tau_{n-1}, \gamma(\tau_{n-1})) \cdots B(\tau_1, \gamma(\tau_1)) \mathcal{D}[\gamma]. \end{aligned}$$

Theorem 5. (Semiclassical approximation). Let T be sufficiently small. Let $F[\gamma] \in \mathcal{F}^\infty$ and the domain of $F[\gamma]$ be continuously extended to $C([0, T] \rightarrow \mathbf{R}^d)$ with respect to the norm $\|\gamma\| = \max_{0 \leq t \leq T} |\gamma(t)|$. Let γ^{cl} be the classical path with $\gamma^{\text{cl}}(0) = x_0$ and $\gamma^{\text{cl}}(T) = x$, and $D(T, x, x_0)$ be the Morette-Van Vleck determinant.

Define $\Upsilon(\hbar, T, x, x_0)$ by

$$\begin{aligned} & \int e^{iS[\gamma]/\hbar} F[\gamma] \mathcal{D}[\gamma] \\ &= \left(\frac{1}{2\pi i \hbar T} \right)^{d/2} e^{iS[\gamma^{\text{cl}}]/\hbar} \left(D(T, x, x_0)^{-1/2} F[\gamma^{\text{cl}}] + \hbar \Upsilon(\hbar, T, x, x_0) \right). \end{aligned}$$

Then, for any multi-indices α, β , there exists a positive constant $C_{\alpha, \beta}$ such that

$$|\partial_x^\alpha \partial_{x_0}^\beta \Upsilon(\hbar, T, x, x_0)| \leq C_{\alpha, \beta} (1 + |x| + |x_0|)^m.$$

Remark. If $\hbar \rightarrow 0$, the remainder term $\hbar \Upsilon(\hbar, T, x, x_0) \rightarrow 0$.

Theorem 6. (New curvilinear integrals along path on path space). Let $0 \leq T' \leq T'' \leq T$. Let m be non-negative integer. Let $Z(t, x)$ be a vector-valued function of $(t, x) \in \mathbf{R} \times \mathbf{R}^d$ into \mathbf{R}^d such that, for any multi-index α , $\partial_x^\alpha Z(t, x)$ and $\partial_x^\alpha \partial_t Z(t, x)$ are continuous on $[0, T] \times \mathbf{R}^d$, and there exists a positive constant C_α such that

$$|\partial_x^\alpha Z(t, x)| + |\partial_x^\alpha \partial_t Z(t, x)| \leq C_\alpha (1 + |x|)^m$$

and $\partial_x Z(t, x)$ is a symmetric matrix, i.e. ${}^t(\partial_x Z) = \partial_x Z$.

Then the curvilinear integrals along paths of Feynman path integral

$$F[\gamma] = \int_{T'}^{T''} Z(t, \gamma(t)) \cdot d\gamma(t) \in \mathcal{F}^\infty.$$

Here $Z \cdot d\gamma$ is the inner product of Z and $d\gamma$ in \mathbf{R}^d .

Remark. In order to explain the difference with known curvilinear integrals on a path space, please, forgive very rough sketch. As examples of curvilinear integrals for paths on a path space, Itô integral [16] and Stratonovich integral [28] for the Brownian motion $\mathbf{B}(t)$ are successful in stochastic analysis. (cf. P. Malliavin [24]) If we can set $\mathbf{B}(T_j) = x_j$, Itô integral is approximated by initial points, i.e.

$$\int_{T'}^{T''} Z(t, \mathbf{B}(t)) \cdot d\mathbf{B}(t) \approx \sum_j Z(T_{j-1}, x_{j-1}) \cdot (x_j - x_{j-1})$$

and Stratonovich integral is approximated by middle points, i.e.

$$\int_{T'}^{T''} Z(t, \mathbf{B}(t)) \circ d\mathbf{B}(t) \approx \sum_j Z\left(\frac{T_j + T_{j-1}}{2}, \frac{x_j + x_{j-1}}{2}\right) \cdot (x_j - x_{j-1}).$$

And many books about Feynman path integrals use endpoints or middle points. On the other hand, if $\gamma = \gamma_{\Delta_{T,0}}$, our new curvilinear integrals is the classical curvilinear integrals itself along the broken line path $\gamma_{\Delta_{T,0}}$, i.e.

$$\int_{T'}^{T''} Z(t, \gamma_{\Delta_{T,0}}(t)) \cdot d\gamma_{\Delta_{T,0}}(t). \quad (2.8)$$

In other words, Itô integral and Stratonovich integral are limits of Riemann sum. On the other hand, our new integral is a limit of curvilinear integral.

Theorem 7. (Fundamental theorem of calculus). *Let m be non-negative integer and $0 \leq T' \leq T'' \leq T$. $g(t, x)$ is a function of $(t, x) \in \mathbf{R} \times \mathbf{R}^d$ such that $g(t, x)$ and $\partial_t g(t, x)$ satisfy Assumption 3. Let T be sufficiently small. Then we have*

$$\begin{aligned} & \int e^{iS[\gamma]/\hbar} \left(g(T'', \gamma(T'')) - g(T', \gamma(T')) \right) \mathcal{D}[\gamma] \\ &= \int e^{iS[\gamma]/\hbar} \left(\int_{T'}^{T''} (\partial_x g)(t, \gamma(t)) \cdot d\gamma(t) + \int_{T'}^{T''} (\partial_t g)(t, \gamma(t)) dt \right) \mathcal{D}[\gamma]. \end{aligned}$$

Furthermore, for any $F[\gamma] \in \mathcal{F}^\infty$ we have

$$\begin{aligned} & \int e^{iS[\gamma]/\hbar} \left(g(T'', \gamma(T'')) - g(T', \gamma(T')) \right) F[\gamma] \mathcal{D}[\gamma] \\ &= \int e^{iS[\gamma]/\hbar} \left(\int_{T'}^{T''} (\partial_x g)(t, \gamma(t)) \cdot d\gamma(t) + \int_{T'}^{T''} (\partial_t g)(t, \gamma(t)) dt \right) F[\gamma] \mathcal{D}[\gamma]. \end{aligned}$$

Remark. (2.8) is the key of the proof of Theorem 7.

Proof of Theorem 7. By Theorem 3(1) and Theorem 2(1), we have

$$G_1[\gamma] = g(T'', \gamma(T'')) - g(T', \gamma(T')) \in \mathcal{F}^\infty.$$

We note that ${}^t(\partial_x^2 g) = (\partial_x^2 g)$. By Theorem 6, Theorem 3(2) and Theorem 2(1) we have

$$G_2[\gamma] = \int_{T'}^{T''} (\partial_x g)(t, \gamma(t)) \cdot d\gamma(t) + \int_{T'}^{T''} (\partial_t g)(t, \gamma(t)) dt \in \mathcal{F}^\infty.$$

By the fundamental theorem of calculus we have $G_1[\gamma_{\Delta_{T,0}}] = G_2[\gamma_{\Delta_{T,0}}]$ for any broken line path $\gamma_{\Delta_{T,0}}$. By Theorem 1 we get

$$\begin{aligned} & \int e^{iS[\gamma]/\hbar} G_1[\gamma] F[\gamma] \mathcal{D}[\gamma] \\ &= \lim_{|\Delta_{T,0}| \rightarrow 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{iS[\gamma_{\Delta_{T,0}}]/\hbar} G_1[\gamma_{\Delta_{T,0}}] F[\gamma_{\Delta_{T,0}}] \prod_{j=1}^J dx_j \\ &= \lim_{|\Delta_{T,0}| \rightarrow 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{iS[\gamma_{\Delta_{T,0}}]/\hbar} G_2[\gamma_{\Delta_{T,0}}] F[\gamma_{\Delta_{T,0}}] \prod_{j=1}^J dx_j \\ &= \int e^{iS[\gamma]/\hbar} G_2[\gamma] F[\gamma] \mathcal{D}[\gamma]. \quad \square \end{aligned}$$

Theorem 8. For any broken line path $\zeta : [0, T] \rightarrow \mathbf{R}^d$, we have the following:

- (1) $(DS)[\gamma][\zeta] \in \mathcal{F}^\infty$.
- (2) $\exp\left(\frac{i}{\hbar}(S[\gamma + \zeta] - S[\gamma])\right) \in \mathcal{F}^\infty$. (The estimates of Theorems 5, 16, 17 depend on \hbar .)

Theorem 9. (Translation). Let T be sufficiently small. For any $F[\gamma] \in \mathcal{F}^\infty$ and any broken line path $\eta : [0, T] \rightarrow \mathbf{R}^d$,

$$\int_{\gamma(0)=x_0, \gamma(T)=x} e^{iS[\gamma+\eta]/\hbar} F[\gamma+\eta] \mathcal{D}[\gamma] = \int_{\gamma(0)=x_0+\eta(0), \gamma(T)=x+\eta(T)} e^{iS[\gamma]/\hbar} F[\gamma] \mathcal{D}[\gamma].$$

Corollary. (Invariance under translation). Let T be sufficiently small. For any $F[\gamma] \in \mathcal{F}^\infty$ and any broken line path $\eta : [0, T] \rightarrow \mathbf{R}^d$ with $\eta(0) = \eta(T) = 0$,

$$\int_{\gamma(0)=x_0, \gamma(T)=x} e^{iS[\gamma+\eta]/\hbar} F[\gamma+\eta] \mathcal{D}[\gamma] = \int_{\gamma(0)=x_0, \gamma(T)=x} e^{iS[\gamma]/\hbar} F[\gamma] \mathcal{D}[\gamma].$$

Proof of Theorem 9. By Theorem 8(1) and Theorem 2(1), (2) we have

$$\exp\left(\frac{i}{\hbar}(S[\gamma + \eta] - S[\gamma])\right) F[\gamma + \eta] \in \mathcal{F}^\infty.$$

By Theorem 1

$$\begin{aligned} & \int_{\gamma(0)=x_0, \gamma(T)=x} e^{iS[\gamma+\eta]/\hbar} F[\gamma+\eta] \mathcal{D}[\gamma] = \lim_{|\Delta_{T,0}| \rightarrow 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \\ & \quad \times \int_{\mathbf{R}^{dJ}} e^{iS[\gamma_{\Delta_{T,0}}]/\hbar} \cdot e^{i(S[\gamma_{\Delta_{T,0}}+\eta]-S[\gamma_{\Delta_{T,0}}])/\hbar} F[\gamma_{\Delta_{T,0}} + \eta] \prod_{j=1}^J dx_j \end{aligned}$$

exists. Choose $\Delta_{T,0}$ which contains all time where the broken line path η breaks. Set $\eta(T_j) = y_j$, $j = 0, 1, \dots, J, J+1$. Since $\gamma_{\Delta_{T,0}} + \eta$ is the broken line path which connects $(T_j, x_j + y_j)$ and $(T_{j-1}, x_{j-1} + y_{j-1})$ by a line segment for $j = 1, 2, \dots, J, J+1$, we can write

$$\begin{aligned} & \int_{\gamma(0)=x_0, \gamma(T)=x} e^{iS[\gamma+\eta]/\hbar} F[\gamma+\eta] \mathcal{D}[\gamma] \\ &= \lim_{|\Delta_{T,0}| \rightarrow 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{iS_{\Delta_{T,0}}(x_{J+1}+y_{J+1}, x_J+y_J, \dots, x_1+y_1, x_0+y_0)/\hbar} \\ & \quad \times F_{\Delta_{T,0}}(x_{J+1}+y_{J+1}, x_J+y_J, \dots, x_1+y_1, x_0+y_0) \prod_{j=1}^J dx_j. \end{aligned}$$

By the change of variables: $x_j + y_j \rightarrow x_j$, $j = 1, 2, \dots, J$, we have

$$\begin{aligned} & \int_{\gamma(0)=x_0, \gamma(T)=x} e^{iS[\gamma+\eta]/\hbar} F[\gamma+\eta] \mathcal{D}[\gamma] \\ &= \lim_{|\Delta_{T,0}| \rightarrow 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{iS_{\Delta_{T,0}}(x_{J+1}+y_{J+1}, x_J, \dots, x_1, x_0+y_0)/\hbar} \\ & \quad \times F_{\Delta_{T,0}}(x_{J+1}+y_{J+1}, x_J, \dots, x_1, x_0+y_0) \prod_{j=1}^J dx_j \\ &= \int_{\gamma(0)=x_0+\eta(0), \gamma(T)=x+\eta(T)} e^{iS[\gamma]/\hbar} F[\gamma] \mathcal{D}[\gamma]. \quad \square \end{aligned}$$

Theorem 10. (Taylor's expansion formula). *Let T be sufficiently small. For any $F[\gamma] \in \mathcal{F}^\infty$ and any broken line path $\eta : [0, T] \rightarrow \mathbf{R}^d$,*

$$\begin{aligned} \int e^{iS[\gamma]/\hbar} F[\gamma+\eta] \mathcal{D}[\gamma] &= \sum_{l=0}^L \frac{1}{l!} \int e^{iS[\gamma]/\hbar} (D^l F)[\gamma][\eta] \cdots [\eta] \mathcal{D}[\gamma] \\ & \quad + \int_0^1 \frac{(1-\theta)^L}{L!} \int e^{iS[\gamma]/\hbar} (D^{L+1} F)[\gamma+\theta\eta][\eta] \cdots [\eta] \mathcal{D}[\gamma] d\theta. \end{aligned} \quad (2.9)$$

Proof of Theorem 10. Using Taylor's expansion formula of (2.4) with respect to $0 \leq \theta \leq 1$ we have

$$F[\gamma+\eta] - \sum_{l=0}^L \frac{1}{l!} (D^l F)[\gamma][\eta] \cdots [\eta] = \int_0^1 \frac{(1-\theta)^L}{L!} (D^{L+1} F)[\gamma+\theta\eta][\eta] \cdots [\eta] d\theta$$

for any broken line path γ . By (2.3) we get (2.9). \square

Theorem 11. (Integration by parts). *Let T be sufficiently small. For any $F[\gamma] \in \mathcal{F}^\infty$ and any broken line path $\eta : [0, T] \rightarrow \mathbf{R}^d$ with $\eta(0) = \eta(T) = 0$,*

$$\int e^{iS[\gamma]/\hbar} (DF)[\gamma][\eta] \mathcal{D}[\gamma] = -\frac{i}{\hbar} \int e^{iS[\gamma]/\hbar} (DS)[\gamma][\eta] F[\gamma] \mathcal{D}[\gamma].$$

Proof of Theorem 11. Choose $\Delta_{T,0}$ which contains all time where the broken line path η breaks. Set $\gamma_{\Delta_{T,0}}(T_j) = x_j$ and $\eta(T_j) = y_j$, $j = 0, 1, \dots, J, J+1$. By Theorem 9 with $\eta(0) = \eta(T) = 0$ we have

$$\begin{aligned}
 0 &= \lim_{|\Delta_{T,0}| \rightarrow 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \\
 &\quad \times \int_{\mathbf{R}^{dJ}} \left(e^{iS[\gamma_{\Delta_{T,0}} + \eta]/\hbar} F[\gamma_{\Delta_{T,0}} + \eta] - e^{iS[\gamma_{\Delta_{T,0}}]/\hbar} F[\gamma_{\Delta_{T,0}}] \right) \prod_{j=1}^J dx_j \\
 &= \lim_{|\Delta_{T,0}| \rightarrow 0} \int_0^1 \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{iS[\gamma_{\Delta_{T,0}} + \theta\eta]/\hbar} \\
 &\quad \times \left(\frac{i}{\hbar} (DS)[\gamma_{\Delta_{T,0}} + \theta\eta][\eta] F[\gamma_{\Delta_{T,0}} + \theta\eta] + (DF)[\gamma_{\Delta_{T,0}} + \theta\eta][\eta] \right) \prod_{j=1}^J dx_j d\theta.
 \end{aligned}$$

Note (2.4) and $y_0 = y_{J+1} = 0$. By the change of variables: $x_j + \theta y_j \rightarrow x_j$, $j = 1, 2, \dots, J$, we have

$$\begin{aligned}
 0 &= \lim_{|\Delta_{T,0}| \rightarrow 0} \int_0^1 \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{iS[\gamma_{\Delta_{T,0}}]/\hbar} \\
 &\quad \times \left(\frac{i}{\hbar} (DS)[\gamma_{\Delta_{T,0}}][\eta] F[\gamma_{\Delta_{T,0}}] + (DF)[\gamma_{\Delta_{T,0}}][\eta] \right) \prod_{j=1}^J dx_j d\theta \\
 &= \frac{i}{\hbar} \int e^{iS[\gamma]/\hbar} (DS)[\gamma][\eta] F[\gamma] \mathcal{D}[\gamma] + \int e^{iS[\gamma]/\hbar} (DF)[\gamma][\eta] \mathcal{D}[\gamma]. \quad \square
 \end{aligned}$$

Theorem 12. (Orthogonal transformation). *Let T be sufficiently small. For any $F[\gamma] \in \mathcal{F}^\infty$ and any $d \times d$ orthogonal matrix Q ,*

$$\int_{\gamma(0)=x_0, \gamma(T)=x} e^{iS[Q\gamma]/\hbar} F[Q\gamma] \mathcal{D}[\gamma] = \int_{\gamma(0)=Qx_0, \gamma(T)=Qx} e^{iS[\gamma]/\hbar} F[\gamma] \mathcal{D}[\gamma].$$

Corollary. (Invariance under orthogonal transformation). *Let T be sufficiently small. For any $F[\gamma] \in \mathcal{F}^\infty$, any $d \times d$ orthogonal matrix Q and any broken line path $\eta : [0, T] \rightarrow \mathbf{R}^d$,*

$$\int_{\gamma(0)=0, \gamma(T)=0} e^{iS[Q\gamma + \eta]/\hbar} F[Q\gamma + \eta] \mathcal{D}[\gamma] = \int_{\gamma(0)=\eta(0), \gamma(T)=\eta(T)} e^{iS[\gamma]/\hbar} F[\gamma] \mathcal{D}[\gamma].$$

Proof of Theorem 12. Since Q is an orthogonal matrix, we can write

$$S[Q\gamma] = \int_0^T \left(\frac{1}{2} \left| \frac{d\gamma}{dt} \right|^2 - V(t, Q\gamma(t)) \right) dt.$$

For any integer $k \geq 2$, there exists a positive constant A'_k such that $|\partial_x^\alpha V(t, Qx)| \leq A'_k$ for any multi-index α with $|\alpha| = k$. By Theorem 1

$$\begin{aligned} & \int_{\gamma(0)=x_0, \gamma(T)=x} e^{iS[Q\gamma]/\hbar} F[Q\gamma] \mathcal{D}[\gamma] \\ &= \lim_{|\Delta_{T,0}| \rightarrow 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{iS[Q\gamma_{\Delta_{T,0}}]/\hbar} F[Q\gamma_{\Delta_{T,0}}] \prod_{j=1}^J dx_j \end{aligned}$$

exists. $Q\gamma_{\Delta_{T,0}}$ is the broken line path which connects (T_j, Qx_j) and (T_{j-1}, Qx_{j-1}) by a line segment for $j = 1, 2, \dots, J, J+1$. By the change of variables: $Qx_j \rightarrow x_j$, $j = 1, 2, \dots, J$ and $|\det Q| = 1$, we have

$$\begin{aligned} & \int_{\gamma(0)=x_0, \gamma(T)=x} e^{iS[Q\gamma]/\hbar} F[Q\gamma] \mathcal{D}[\gamma] \\ &= \lim_{|\Delta_{T,0}| \rightarrow 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{iS_{\Delta_{T,0}}(Qx_{J+1}, x_J, \dots, x_1, Qx_0)/\hbar} \\ & \quad \times F_{\Delta_{T,0}}(Qx_{J+1}, x_J, \dots, x_1, Qx_0) \prod_{j=1}^J dx_j \\ &= \int_{\gamma(0)=Qx_0, \gamma(T)=Qx} e^{iS[\gamma]/\hbar} F[\gamma] \mathcal{D}[\gamma]. \quad \square \end{aligned}$$

3 The details about the convergence in Theorem 1

For simplicity, we set

$$\begin{aligned} x_{L,l} &= (x_L, x_{L-1}, \dots, x_l) & \text{for any } 0 \leq l \leq L \leq J+1, \\ m_+ &= \max(m, 0) & \text{for any integer } m. \end{aligned}$$

Let T satisfy $4A_2dT^2 < 1$ where A_2 is a constant in Assumption 1. Then we can define $x_{J,1}^\dagger = x_{J,1}^\dagger(x_{J+1}, x_0)$ by

$$(\partial_{x_{J,1}} S_{\Delta_{T,0}})(x_{J+1}, x_{J,1}^\dagger, x_0) = 0.$$

For any given function $f = f(x_{J+1}, x_{J,1}, x_0)$ let f^\dagger be the function obtained by pushing $x_{J,1} = x_{J,1}^\dagger$ into f , i.e.

$$f^\dagger = f^\dagger(x_{J+1}, x_0) = f(x_{J+1}, x_{J,1}^\dagger, x_0).$$

Then we have

$$\begin{aligned} \gamma_{\Delta_{T,0}}^\dagger &= \gamma_{\Delta_{T,0}}(t, x_{J+1}, x_{J,1}^\dagger, x_0), \\ S[\gamma_{\Delta_{T,0}}^\dagger] &= S_{\Delta_{T,0}}^\dagger(x_{J+1}, x_0) = S_{\Delta_{T,0}}(x_{J+1}, x_{J,1}^\dagger, x_0), \\ F[\gamma_{\Delta_{T,0}}^\dagger] &= F_{\Delta_{T,0}}^\dagger(x_{J+1}, x_0) = F_{\Delta_{T,0}}(x_{J+1}, x_{J,1}^\dagger, x_0). \end{aligned}$$

We define $D_{\Delta_{T,0}}(x_{J+1}, x_0)$ by

$$D_{\Delta_{T,0}}(x_{J+1}, x_0) = \det \left(\partial_{x_{J,1}}^2 S_{\Delta_{T,0}} \right)^\dagger \times \left(\frac{\prod_{j=1}^{J+1} t_j}{T_{J+1}} \right)^d.$$

Furthermore we define the remainder term $\Upsilon_{\Delta_{T,0}}(\hbar, x_{J+1}, x_0)$ by

$$\begin{aligned} & \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{iS[\gamma_{\Delta_{T,0}}]/\hbar} F[\gamma_{\Delta_{T,0}}] \prod_{j=1}^J dx_j \\ &= \left(\frac{1}{2\pi i \hbar T} \right)^{d/2} e^{iS[\gamma_{\Delta_{T,0}}^\dagger]/\hbar} \left(D_{\Delta_{T,0}}(x, x_0)^{-1/2} F[\gamma_{\Delta_{T,0}}^\dagger] + \hbar \Upsilon_{\Delta_{T,0}}(\hbar, x, x_0) \right). \end{aligned}$$

We can prove the convergence of (2.3) in the following order.

Theorem 13. (Convergence of Path). *Let $4A_2 dT^2 < \frac{1}{2}$. Then, for any multi-indices α, β , there exist positive constants $C_{\alpha,\beta}, C'_{\alpha,\beta}$ independent of $\Delta_{T,0}$ such that*

$$\begin{aligned} \left\| \partial_x^\alpha \partial_{x_0}^\beta \gamma_{\Delta_{T,0}}^\dagger \right\| &\leq C_{\alpha,\beta} (1 + |x| + |x_0|)^{(1-|\alpha+\beta|)_+}, \\ \left\| \partial_x^\alpha \partial_{x_0}^\beta \left(\gamma_{\Delta_{T,0}}^\dagger - \gamma^{\text{cl}} \right) \right\| &\leq C'_{\alpha,\beta} |\Delta_{T,0}| T (1 + |x| + |x_0|), \end{aligned}$$

where $\gamma^{\text{cl}} = \gamma^{\text{cl}}(t, x, x_0)$ is the classical path with $\gamma^{\text{cl}}(0) = x_0$ and $\gamma^{\text{cl}}(T) = x$, and $\|\gamma\| = \max_{0 \leq t \leq T} |\gamma(t)|$.

Theorem 14. (Convergence of Phase). *Let $4A_2 dT^2 < \frac{1}{2}$. Then, for any multi-indices α, β , there exist positive constants $C_{\alpha,\beta}, C'_{\alpha,\beta}$ independent of $\Delta_{T,0}$ such that*

$$\begin{aligned} \left| \partial_x^\alpha \partial_{x_0}^\beta \left(S_{\Delta_{T,0}}^\dagger(x, x_0) - \frac{(x - x_0)^2}{2T} \right) \right| &\leq C_{\alpha,\beta} T (1 + |x| + |x_0|)^{(2-|\alpha+\beta|)_+}, \\ \left| \partial_x^\alpha \partial_{x_0}^\beta \left(S_{\Delta_{T,0}}^\dagger(x, x_0) - S(T, x, x_0) \right) \right| &\leq C'_{\alpha,\beta} |\Delta_{T,0}|^2 T (1 + |x| + |x_0|)^{1+(1-|\alpha+\beta|)_+}, \end{aligned}$$

where $S(T, x, x_0) = S[\gamma^{\text{cl}}]$ is the action along the classical path γ^{cl} with $\gamma^{\text{cl}}(0) = x_0$ and $\gamma^{\text{cl}}(T) = x$.

Theorem 15. (Convergence of Main term 1). *Let T be sufficiently small. Then, for any multi-indices α, β , there exist positive constants $C_{\alpha,\beta}, C'_{\alpha,\beta}$ independent of $\Delta_{T,0}$ such that*

$$\begin{aligned} \left| \partial_x^\alpha \partial_{x_0}^\beta (D_{\Delta_{T,0}}(x, x_0) - 1) \right| &\leq C_{\alpha,\beta} T^2, \\ \left| \partial_x^\alpha \partial_{x_0}^\beta (D_{\Delta_{T,0}}(x, x_0) - D(T, x, x_0)) \right| &\leq C'_{\alpha,\beta} |\Delta_{T,0}| T (1 + |x| + |x_0|), \end{aligned}$$

where $D(T, x, x_0)$ is the Morette–Van Vleck determinant.

Theorem 16. (Convergence of Main term 2). *Let T be sufficiently small. Let $F[\gamma] \in \mathcal{F}$. Then, for any multi-indices α, β , there exist positive constants $C_{\alpha,\beta}, C'_{\alpha,\beta}$ independent of $\Delta_{T,0}$ such that*

$$\begin{aligned} \left| \partial_x^\alpha \partial_{x_0}^\beta F_{\Delta_{T,0}}^\dagger(x, x_0) \right| &\leq C_{\alpha,\beta} (1 + |x| + |x_0|)^m, \\ \left| \partial_x^\alpha \partial_{x_0}^\beta \left(F_{\Delta_{T,0}}^\dagger(x, x_0) - F(T, x, x_0) \right) \right| &\leq C'_{\alpha,\beta} |\Delta_{T,0}| T (1 + |x| + |x_0|)^{m+1}, \end{aligned}$$

with a function $F(T, x, x_0)$. Furthermore, if the domain of $F[\gamma]$ is continuously extended to $C([0, T] \rightarrow \mathbf{R}^d)$ with respect to the norm $\|\gamma\| = \max_{0 \leq t \leq T} |\gamma(t)|$, then $F(T, x, x_0) = F[\gamma^{\text{cl}}]$.

Theorem 17. (Convergence of Remainder term). *Let T be sufficiently small. Let $F[\gamma] \in \mathcal{F}$. Then, for any multi-indices α, β , there exist positive constants $C_{\alpha,\beta}, C'_{\alpha,\beta}$ independent of $\Delta_{T,0}$ and \hbar such that*

$$\begin{aligned} \left| \partial_x^\alpha \partial_{x_0}^\beta \Upsilon_{\Delta_{T,0}}(\hbar, x, x_0) \right| &\leq C_{\alpha,\beta} T (U + T^2) (1 + |x| + |x_0|)^m, \\ \left| \partial_x^\alpha \partial_{x_0}^\beta \left(\Upsilon_{\Delta_{T,0}}(\hbar, x, x_0) - \Upsilon(\hbar, T, x, x_0) \right) \right| &\leq C'_{\alpha,\beta} |\Delta_{T,0}| (U + T^2) (1 + |x| + |x_0|)^{m+1} \end{aligned}$$

with a function $\Upsilon(\hbar, T, x, x_0)$.

4 Assumption 2' by 'piecewise classical paths'

As a remark on Assumption 2, we state Assumption 2' under the time slicing approximation by 'piecewise classical paths'.

Let γ^{cl} be the classical path with $\gamma^{\text{cl}}(0) = x_0$ and $\gamma^{\text{cl}}(T) = x$, i.e. γ^{cl} satisfies the Euler's equation

$$\frac{d^2}{dt^2} \gamma^{\text{cl}}(t) - (\partial_x V)(t, \gamma^{\text{cl}}(t)) = 0.$$

For any division $\Delta_{T,0} : T = T_{J+1} > T_J > \dots > T_1 > T_0 = 0$, let

$$\gamma_{\Delta_{T,0}} = \gamma_{\Delta_{T,0}}(t, x_{J+1}, x_J, \dots, x_1, x_0)$$

be the 'piecewise classical path' which connects (T_j, x_j) and (T_{j-1}, x_{j-1}) by a classical path for any $j = 1, 2, \dots, J+1$. Set

$$\begin{aligned} S[\gamma_{\Delta_{T,0}}] &= S_{\Delta_{T,0}}(x_{J+1}, x_J, \dots, x_1, x_0), \\ F[\gamma_{\Delta_{T,0}}] &= F_{\Delta_{T,0}}(x_{J+1}, x_J, \dots, x_1, x_0). \end{aligned}$$

Assumption 2' Let m and U be non-negative integers and $u_j, j = 1, 2, \dots, J, J+1$ be non-negative parameters depending on $\Delta_{T,0}$ such that $\sum_{j=1}^{J+1} u_j \leq U < \infty$. For any non-negative integer M , there exists a positive constant C_M such that for any division $\Delta_{T,0}$, any $|\alpha_j| \leq M, j = 0, 1, \dots, J, J+1$, and any $1 \leq k \leq J$,

$$\begin{aligned} \left| \left(\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j} \right) F_{\Delta_{T,0}}(x_{J+1}, x_J, \dots, x_1, x_0) \right| &\leq (C_M)^{J+1} \left(1 + \sum_{j=0}^{J+1} |x_j| \right)^m, \\ \left| \left(\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j} \right) \partial_{x_k} F_{\Delta_{T,0}}(x_{J+1}, x_J, \dots, x_1, x_0) \right| &\leq (C_M)^{J+1} (u_{k+1} + u_k) \left(1 + \sum_{j=0}^{J+1} |x_j| \right)^m. \end{aligned}$$

The class \mathcal{F}' of functionals $F[\gamma]$ defined by Assumption 2' also satisfies Theorem 1, Theorem 2(1) and Theorems 3, 4, 5, 6, 7. Furthermore, the convergence of the time slicing approximation by 'piecewise classical paths' is much sharper than the convergence of the time slicing approximation by broken line paths. For the details, see D. Fujiwara and N. Kumano-go [11, 12]. In [12], D. Fujiwara wrote down the second term of the semi-classical asymptotic expansion of Feynman path integrals with the integrand $F[\gamma]$ of polynomial growth.

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